Analysis of $p$-Adic Numbers

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Setup

- $p$ any prime, $p = 2, 3, 5, \ldots$, $\gamma \in \mathbb{Z}, \mathbb{Q}$, let $\mathbb{Z}_+$ be the natural numbers. If $K$ is a field, then $K^\times$ is the multiplicative group.
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$$x = \pm p^\gamma \frac{a}{b},$$

for $\gamma \in \mathbb{Z}$, $a, b \in \mathbb{Z}_+$, with $a, b$ not divisible by $p$ and $\gcd(a, b) = 1$
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- **p**-adic norm: \( |x|_p \) of \( x \in \mathbb{Q} \) is \( |x|_p = p^{-\gamma} \) for \( x \neq 0 \), \( |0|_p := 0 \)
- \( \mathbb{Q}_p \): field of \( p \)-adic numbers, is the completion of the field \( \mathbb{Q} \) w.r.t. the norm \(| \cdot |_p\)
Every nonzero $p$-adic number, can be written as

$$x = \sum_{i=0}^{\infty} x_i p^i,$$

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$-\gamma$ is called the order of $x$ and denoted $\text{ord } x = -\gamma$ and $\text{ord } 0 := -\infty$
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If $x$ a nonzero integer, then

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Norm Properties

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Means the norm in \(\mathbb{Q}_p\) is an ultrametric
Topology of $\mathbb{Q}_p$

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- Have the following relations:

$$B_\gamma(a) = \bigcup_{\gamma' \leq \gamma} S_{\gamma'}(a), \quad S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a)$$

$$\mathbb{Q}_p = \bigcup_{\gamma \in \mathbb{Z}} B_\gamma(a), \quad \mathbb{Q}_p^\times = \bigcup_{\gamma \in \mathbb{Z}} S_\gamma(a)$$
Fun facts about the space $\mathbb{Q}_p$

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- A disk is open and compact
Some Calculus of $\mathbb{Q}_p$

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\int_{Z_p} d_p x = 1,
\int_{B_0} d_p x = p^\gamma,
\int_{S^\infty} d_p x = \left(1 - \frac{1}{p^\alpha}\right)p^\gamma,
\int_{\mathbb{Q}_p} f(x) d_p x = \sum_{\gamma = -\infty}^{\infty} \int_{\mathbb{S}_\gamma} f(x) d_p x,
\int_{B_0} |x|_p^{\alpha - 1} d_p x = 1 - \frac{1}{p^\alpha}.
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- $d_p x$ has the following properties:
  1. $d_p (x + a) = d_p x$ for $a \in \mathbb{Q}_p$, 2. $d_p (ax) = |a|_p d_p x$ for $a \in \mathbb{Q}_p^\times$
The \( p \)-Adic Numbers

Some Analysis of \( p \)-adic numbers

Possible Research Problems

References

Some Calculus of \( \mathbb{Q}_p \)

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- Normally have \( dx \) in regular calculus, here we have \( d_p x \).
- \( d_p x \) has the following properties:
  1. \( d_p(x + a) = d_p x \) for \( a \in \mathbb{Q}_p \)
  2. \( d_p(ax) = |a|_p d_p x \) for \( a \in \mathbb{Q}_p^\times \)
- We have the following integrals:
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  \int_{\mathbb{Z}_p} d_p x = 1, \quad \int_{B_\gamma} d_p x = p^\gamma, \quad \int_{S_\gamma} d_p x = \left(1 - \frac{1}{p}\right) p^\gamma
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  \[
  \int_{\mathbb{Q}_p} f(x) d_p x = \sum_{\gamma = -\infty}^{\infty} \int_{S_\gamma} f(x) d_p x, \quad \int_{B_\gamma} |x|_p^{\alpha - 1} = \frac{1 - p^{-1}}{1 - p^{-\alpha}} p^{\alpha \gamma}, \alpha > 0
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Differential Operators

Let $L$ be a differential operator, we can just think $L = \frac{d}{dx}$, and let $\Omega$ be some domain.
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\begin{aligned}
&Lu = f \text{ in } \Omega \\
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\]

- The goal is to find functions $u$ that satisfy this type of boundary value problem.
Another way to rephrase the goal, is one wants to find, if possible, $L^{-1}$ or $(L - \lambda)^{-1}$ for complex numbers $\lambda$. 

This has been studied very extensively classically, meaning when $L = \frac{d}{dx}$ or $L = a_n(x) \frac{d^n}{dx^n} + \cdots + a_1(x) \frac{d}{dx} + a_0(x)$, or $L$ involve partial derivatives etc. These types of questions will lead to problems in ODEs or PDEs. Current research still goes on in this type of question.
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Relation to $p$-adic Analysis

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- Now $L$ becomes some kind of difference of the vertices and edges making it seem like the problem becomes a discrete problem.
- Now one has to study function spaces of sequences and series that relate to $\mathbb{Q}_p$
References

The End

Thank You