MA 1723 Practice Exam 2 Solutions

1.) want to know x for $f(x)=2 \Rightarrow 2 = x^3+3\sin x+2\cos x \Rightarrow x=0$. Then $f'(x)=3x^2+3\cos x-2\sin x$ and $f'(0)=3$

$\therefore (f^{-1})'(2) = \frac{1}{3}$

2.) $g(x)=(f^{-1}(x))^{-1} \therefore g'(x) = -(f^{-1}(x))^{-2}(f^{-1})'(x) = \frac{-1}{(f^{-1}(x))^2}\frac{1}{f'(f^{-1}(x))}$. since $f(3)=2 \Rightarrow f^{-1}(2)=3$

$\Rightarrow g'(2) = \frac{-1}{(f^{-1}(2))^2}\frac{1}{f'(3)} = -\frac{1}{9}\cdot\frac{1}{\frac{1}{9}} = -1$

3.) $y=x^{\sin x}$ so $\ln y = \sin x \ln x \Rightarrow \frac{y'}{y} = \cos x\ln x+\frac{\sin x}{x} \Rightarrow y'=y\left(\cos x\ln x+\frac{\sin x}{x}\right) \Rightarrow y'=x^{\sin x}\left(\cos x\ln x+\frac{\sin x}{x}\right)$

4.) let $y=(\sin x)^{\ln x}$ so $\ln y = \ln x\ln(\sin x) \Rightarrow \frac{y'}{y} = \frac{1}{x}\ln(\sin x)+\ln x\frac{\cos x}{\sin x} = \frac{1}{x}\ln(\sin x)+\ln x\cot x$

$\Rightarrow y'=(\sin x)^{\ln x}\left(\frac{\ln(\sin x)}{x}+\ln x\cot x\right)$

5.) $\int \frac{1+x}{1+x^2}dx = \int\frac{1}{1+x^2}dx+\int\frac{x}{1+x^2}dx \leftarrow \begin{matrix} u=1+x^2 \\ du=2x\,dx\end{matrix}$ $\therefore \int\frac{1+x}{1+x^2}dx = \tan^{-1}x+\frac{1}{2}\int\frac{1}{u}du = \tan^{-1}x+\frac{1}{2}\ln u+C$

$= \tan^{-1}x+\frac{1}{2}\ln(x^2+1)+C$

6.) $\int\frac{dx}{\sin^{-1}x\sqrt{1-x^2}}$ $\begin{matrix} u=\sin^{-1}x \\ du=\frac{1}{\sqrt{1-x^2}}dx\end{matrix}$ $\therefore \int\frac{dx}{\sin^{-1}x\sqrt{1-x^2}} = \int\frac{du}{u} = \ln|u|+C = \ln|\sin^{-1}x|+C$

7.) let $y=\lim_{x\to0^+}x^{\sqrt{x}} \Rightarrow \ln y = \lim_{x\to0^+}\sqrt{x}\ln x \overset{0\cdot\infty \text{ type}}{=} \lim_{x\to0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}} \overset{\frac{\infty}{\infty}\text{ L'H}}{=} \lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{2}\frac{1}{x^{3/2}}} = -2\lim_{x\to0^+}x^{\frac{1}{2}} = 0$

$\therefore y=e^0=1$

8.) let $y=\lim_{x\to\infty}x^{\frac{1}{x}} \Rightarrow \ln y = \lim_{x\to\infty}\frac{1}{x}\ln x = \lim_{x\to\infty}\frac{\ln x}{x} \overset{\frac{\infty}{\infty}\text{-type L'H}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0 \quad \therefore y=e^0=1$

9.) $\int e^{\sqrt{x}}dx$ let $w=\sqrt{x}$ so $dw=\frac{1}{2\sqrt{x}}dx \Rightarrow dx=2w\,dw \Rightarrow \int e^{\sqrt{x}}dx = 2\int we^w dw$ let $\begin{matrix} u=w & dv=e^w dw \\ du=dw & v=e^w\end{matrix}$

so $2\int we^w dw = 2\left(we^w-\int e^w dw\right) = 2(we^w-e^w)+C = 2(\sqrt{x}-1)e^{\sqrt{x}}+C$

10.) $\int x\ln(x+1)dx$ let $w=\ln(x+1)$ so $dw=\frac{1}{x+1}dx \Rightarrow e^w dw=dx$ so $\int x\ln(x+1)dx = \int(e^w-1)e^w w\,dw = \int we^{2w}dw - \int we^w dw$

$x+1=e^w \Rightarrow x=e^w-1$

$\begin{matrix} u=w \\ dv=e^{2w}dw\end{matrix}$ $\quad\begin{matrix} u=w \\ dv=e^w dw\end{matrix}$

so $\int x\ln(x+1)dx = \left(\frac{1}{2}we^{2w}-\frac{1}{2}\int e^{2w}dw\right)-\left(we^w-\int e^w dw\right) = \frac{1}{2}we^{2w}-\frac{1}{4}e^{2w}-we^w+e^w+C$

$= \frac{1}{2}(x+1)^2\ln(x+1)-\frac{1}{4}(x+1)^2-(x+1)\ln(x+1)+(x+1)+C$

11.) $\int \sin^3\theta \cos^4\theta \, d\theta = \int \sin\theta \sin^2\theta \cos^4\theta \, d\theta = \int \sin\theta (1-\cos^2\theta)\cos^4\theta \, d\theta$ $\quad u = \cos\theta$, $du = -\sin\theta \, d\theta$ $\Rightarrow \int = \int (1-u^2)u^4 \, du =$

$= \int (u^4 - u^6) \, du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\cos^5\theta}{5} - \frac{\cos^7\theta}{7} + C$

12.) $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \left(\frac{\sin(2x)}{2}\right)^2 dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2(2x) \, dx = \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos(4x)) \, dx = \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{16}$

13.) $\int f(x) \, dx$, let $u = f(x)$, $du = f'(x)dx$; $dv = dx$, $v = x$; so $\int f(x)dx = x f(x) - \int x f'(x) dx$. let $y = f(x)$ so $g(y) = f^{-1}(y) = x$

Now $\int_a^b f(x) dx = b f(b) - a f(a) - \int_a^b x f'(x) dx$. ~~let~~ now $y = f(x)$ so $dy = f'(x)dx$ $\Rightarrow \int_a^b x f'(x) dx = \int_{f(a)}^{f(b)} g(y) dy$

so $\int_a^b f(x) dx = b f(b) - a f(a) - \int_{f(a)}^{f(b)} g(y) dy$