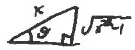
MA 1723 Practice Final Exam Solutions

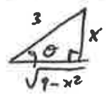


1.) $\int \frac{\sqrt{x^2-1}}{x^4}\,dx$ let $x=\sec\theta$, $dx=\sec\theta\tan\theta\,d\theta$ so $\int \frac{\sqrt{\sec^2\theta-1}}{\sec^4\theta}\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \int \frac{\sec^2\theta-1}{\sec^3\theta}\,d\theta = \int\left(\frac{1}{\sec\theta}-\frac{1}{\sec^3\theta}\right)d\theta$

$= \int \cos\theta\,d\theta - \int \cos^3\theta\,d\theta = \sin\theta - \int(1-\sin^2\theta)\cos\theta\,d\theta$ $\quad u=\sin\theta$, $du=\cos\theta\,d\theta \Rightarrow \int = \sin\theta - \int(1-u^2)\,du = \sin\theta - \left(u-\frac{u^3}{3}\right)+C$

$= \sin\theta - \sin\theta + \frac{1}{3}\sin^3\theta + C = \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C$

2.) $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ let $x=3\sin\theta$, $dx=3\cos\theta\,d\theta$



so $\int = \int \frac{9\sin^2\theta\,(3\cos\theta)\,d\theta}{3\cos\theta} = 9\int \sin^2\theta\,d\theta = \frac{9}{2}\int(1-\cos(2\theta))\,d\theta$

$= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C = \frac{9}{2}(\theta - \sin\theta\cos\theta) = \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$

3.) $\int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \frac{2}{(2x+1)(x+1)}\,dx = \int_0^1\left(\frac{A}{2x+1}+\frac{B}{x+1}\right)dx$ so need $2 = A(x+1) + B(2x+1)$. $x=-1:\ 2=-B \Rightarrow B=-2$

$x=-\frac{1}{2}:\ 2=\frac{A}{2} \Rightarrow A=4$

So $\int_0^1 = \int_0^1\left(\frac{4}{2x+1} - \frac{2}{x+1}\right)dx = \frac{1}{2}\cdot 4\ln(2x+1)\Big|_0^1 - 2\ln(x+1)\Big|_0^1 = 2\ln 3 - 2\ln 2 = 2\ln\left(\frac{3}{2}\right)$

4.) $\int_{-1}^0 \frac{x^3-4x+1}{x^2-3x+2}\,dx$ 1st long divide

$$\begin{array}{r} x+3 \\ x^2-3x+2\,\overline{)\,x^3-4x+1} \\ -x^3+3x^2-2x \\ \hline 3x^2-6x+1 \\ -3x^2+9x-6 \\ \hline 3x-5 \end{array}$$

so $\frac{x^3-4x+1}{x^2-3x+2} = x+3 + \frac{3x-5}{x^2-3x+2} = x+3 + \frac{3x-5}{(x-2)(x-1)} = x+3 + \frac{A}{x-2} + \frac{B}{x-1}$

$\Rightarrow 3x-5 = A(x-1) + B(x-2)$

$x=1:\ -2=-B \Rightarrow B=2$

$x=2:\ 1=A$

So $\int_{-1}^0 = \int_{-1}^0 (x+3)\,dx + \int_{-1}^0\left(\frac{1}{x-2}+\frac{2}{x-1}\right)dx = \left(\frac{x^2}{2}+3x\right)\Big|_{-1}^0 + \ln|x-2|\Big|_{-1}^0 + 2\ln|x-1|\Big|_{-1}^0 = -\frac{1}{2}+3 + \ln 2 - \ln 3 - 2\ln 2$

$= \frac{5}{2} - \ln 2 - \ln 3 = \frac{5}{2} - \ln 6$

5.) $\int_0^\infty e^{-\sqrt{x}}\,dx$ let $w=\sqrt{x}$, $dw = \frac{1}{2\sqrt{x}}\,dx$ $\Rightarrow 2w\,dw = dx$; $x=0 \Rightarrow w=0$; as $x\to\infty \Rightarrow w\to\infty$ so $\int = 2\int_0^\infty w e^{-w}\,dw = 2\lim_{b\to\infty}\int_0^b w e^{-w}\,dw$ $\quad u=w$, $du=dw$; $dv=e^{-w}dw$, $v=-e^{-w}$

$\Rightarrow \int = 2\lim_{b\to\infty}\left(-we^{-w}\Big|_0^b + \int_0^b e^{-w}\,dw\right) = 2\lim_{b\to\infty}\left(-be^{-b} + 1 - e^{-b}\right) = 2$ by L'H.

6.) $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$ let $u=\ln x$, $du=\frac{1}{x}\,dx$; $x=e \Rightarrow u=1$; $x\to\infty \Rightarrow u\to\infty$ so $\int = \int_1^\infty \frac{1}{u^2}\,du = \lim_{b\to\infty}\int_1^b \frac{1}{u^2}\,du = \lim_{b\to\infty} -\frac{1}{u}\Big|_1^b = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$

7.) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to1^-}\sin^{-1}x\Big|_0^b = \lim_{b\to1^-}\sin^{-1}b = \frac{\pi}{2}$ as $\sin^{-1}y$ is cont. from left on

---

8.) $\int_0^5 \frac{x}{x-2}\,dx = \int_0^5\left(\frac{x-2}{x-2}+\frac{2}{x-2}\right)dx = \int_0^5 1\,dx + \int_0^5 \frac{2}{x-2}\,dx$. Focus on $\int_0^5 \frac{2}{x-2}\,dx = \int_0^2 \frac{2}{x-2}\,dx + \int_2^5 \frac{2}{x-2}\,dx$

$= \lim_{b\to2^-}\int_0^b \frac{2}{x-2}\,dx + \lim_{b\to2^+}\int_b^5 \frac{2}{x-2}\,dx = \lim_{b\to2^-} 2\ln|x-2|\Big|_0^b + \lim_{b\to2^+} 2\ln|x-2|\Big|_b^5$ but neither limit exists.

$\therefore$ the integral diverges.

9.) $y = \ln(\cos x)$ so $y' = \frac{-\sin x}{\cos x}$ and $(y')^2 = \tan^2 x$ then $1+(y')^2 = 1+\tan^2 x = \sec^2 x$

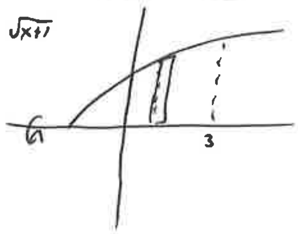
then $L = \int_0^{\frac{\pi}{3}} \sqrt{1+(y')^2}\,dx = \int_0^{\frac{\pi}{3}} \sec x\,dx = \ln(\sec x + \tan x)\Big|_0^{\frac{\pi}{3}} = \ln(2+\sqrt{3})$

10.) $y = \sqrt{x-x^2} + \sin^{-1}(\sqrt{x})$ notice dom of $\sqrt{x-x^2}$ is $0 \le x \le 1$ similarly for $\sin^{-1}(\sqrt{x})$ $\therefore$ limits of int. are $x=0,1$

then $y' = \frac{1}{2\sqrt{x-x^2}}(1-2x) + \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x-x^2}}(1-2x+1) = \frac{1-x}{\sqrt{x-x^2}}$ so $(y')^2 = \frac{(1-x)^2}{x(1-x)} = \frac{1-x}{x}$ and $\sqrt{1+(y')^2} = \sqrt{\frac{x+1-x}{x}} = \frac{1}{\sqrt{x}}$

so $L = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 1^-}\int_0^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 1^-} 2\sqrt{x}\Big|_0^b = \lim_{b\to 1^-} 2\sqrt{b} = 2$

11.) $y^2 = x+1$, $0 \le x \le 3$ so $y = \sqrt{x+1}$
about x-axis

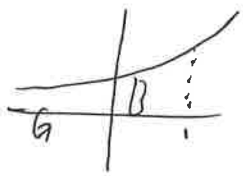


$S \cong 2\pi y\,ds$ so $y' = \frac{1}{2\sqrt{x+1}}$, $(y')^2 = \frac{1}{4(x+1)}$ $1+(y')^2 = \frac{4x+5}{4(x+1)}$

so $S = 2\pi\int_0^3 \sqrt{x+1}\sqrt{\frac{4x+5}{4(x+1)}}\,dx = \pi\int_0^3 \sqrt{4x+5}\,dx$ $u = 4x+5$ $du = 4dx$ $x=0 \Rightarrow u=5$ $x=3 \Rightarrow u=17$

$= \pi\int_5^{17} \sqrt{u}\,du = \frac{2\pi}{3}u\sqrt{u}\Big|_5^{17} = \frac{2\pi}{3}(17\sqrt{17} - 5\sqrt{5})$

12.) $y = \sqrt{1+e^x}$, $0 \le x \le 1$
about x-axis



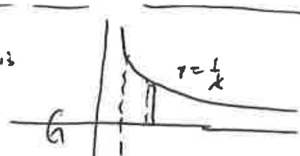
$S \cong 2\pi y\,ds$ so $y' = \frac{e^x}{2\sqrt{1+e^x}}$ and $1+(y')^2 = 1 + \frac{e^{2x}}{4(1+e^x)} = \frac{4+4e^x+e^{2x}}{4(1+e^x)} = \frac{(2+e^x)^2}{4(1+e^x)}$

so $S = 2\pi\int_0^1 \sqrt{1+e^x}\sqrt{\frac{(2+e^x)^2}{4(1+e^x)}}\,dx = \pi\int_0^1 (2+e^x)\,dx = \pi(2x+e^x)\Big|_0^1$

$= \pi(2+e-1) = \pi(e+1)$

---

13) $y = \frac{1}{x}$ about x-axis
$x=1$ and $x \to \infty$



1st vol washer $= \frac{1}{x^2}$ so $V = \pi\int_1^\infty \frac{1}{x^2}\,dx = \pi\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \pi\lim_{b\to\infty} \frac{-1}{x}\Big|_1^b$

$= \pi\lim_{b\to\infty}(1 - \frac{1}{b}) = \pi$

2nd for surface area, $S \cong 2\pi y\,ds$ so $y' = -\frac{1}{x^2}$, and $1+(y')^2 = 1+\frac{1}{x^4} \ge 1$ $\therefore$ $S = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \ge 2\pi\int_1^\infty \frac{1}{x}\,dx$

$= 2\pi\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = 2\pi\lim_{b\to\infty}\ln b \longrightarrow \infty$.