MA 2733 Practice Exam 1 Solutions

1) want $\lim_{n\to\infty}(\ln(n+1)-\ln n) = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln\left(1+\lim_{n\to\infty}\frac{1}{n}\right) = \ln(1) = 0.$

2.) $\lim_{n\to\infty} \frac{(2n)!}{(3n)!}$. notice $(2n)! = 2\cdot 4\cdot 6\cdot 8\cdots(2n-2)2n = 2^n n!$.

$(3n)! = 3\cdot 6\cdot 9\cdot 12\cdots(3n-3)3n = 3^n n!$

so $\lim_{n\to\infty}\frac{(2n)!}{(3n)!} = \lim_{n\to\infty}\frac{2^n n!}{3^n n!} = \lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$ since $\frac{2}{3}<1$.

3.) $\sum_{n=1}^{\infty}\frac{2^n+e^n}{\pi^n} = \sum_{n=1}^{\infty}\frac{2^n}{\pi^n} + \sum_{n=1}^{\infty}\frac{e^n}{\pi^n} = \sum_{n=1}^{\infty}\frac{2}{\pi}\left(\frac{2}{\pi}\right)^{n-1} + \sum_{n=1}^{\infty}\frac{e}{\pi}\left(\frac{e}{\pi}\right)^{n-1} = I + II$

both I and II are geometric w/ $a=\frac{2}{\pi}$, $r=\frac{2}{\pi}$ for I and $a=\frac{e}{\pi}$, $r=\frac{e}{\pi}$ for II

since $\left|\frac{2}{\pi}\right| = \frac{2}{\pi}<1$ and $\left|\frac{e}{\pi}\right| = \frac{e}{\pi}<1$ they both converge so $I = \frac{\frac{2}{\pi}}{1-\frac{2}{\pi}} = \frac{2}{\pi-2}$

and $II = \frac{\frac{e}{\pi}}{1-\frac{e}{\pi}} = \frac{e}{\pi-e}$ so $\sum_{n=1}^{\infty}\frac{2^n+e^n}{\pi^n} = \frac{2}{\pi-2}+\frac{e}{\pi-e}$

4.) let $s_N = \sum_{n=1}^{N}\left(e^{\frac{1}{n}}-e^{\frac{1}{n+1}}\right) = (e-e^{\frac{1}{2}})+(e^{\frac{1}{2}}-e^{\frac{1}{3}})+\cdots+(e^{\frac{1}{N}}-e^{\frac{1}{N+1}}) = e-e^{\frac{1}{N+1}}$

since $\frac{1}{N+1}\to 0$ as $N\to\infty \Rightarrow e^{\frac{1}{N+1}}\to 1$ as $N\to\infty \Rightarrow \sum_{n=1}^{\infty}\left(e^{\frac{1}{n}}-e^{\frac{1}{n+1}}\right) = \lim_{N\to\infty} s_N = e-1$

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5). $\sum_{n=1}^{\infty} ne^{-n}$ consider $f(x) = xe^{-x}$ then $f'(x) = e^{-x}-xe^{-x} = e^{-x}(1-x)\le 0$ for $x\ge 1$

so $f \searrow$. so by int. test $\sum_{n=1}^{\infty} ne^{-n} \le \int_1^{\infty} xe^{-x}\,dx$. let $u=x$, $dv=e^{-x}dx$, $du=dx$, $v=-e^{-x}$

$\Rightarrow \int_1^{\infty} xe^{-x}\,dx = -xe^{-x}\Big|_1^{\infty}+\int_1^{\infty}e^{-x}\,dx = e^{-1}-e^{-x}\Big|_1^{\infty} = 2e^{-1}<\infty$ so series converges.

6.) $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ consider $f(x)=\frac{1}{x\ln x}$ as $x\to\infty$ $\ln x\nearrow\infty$ so does $x\nearrow\infty$

clear $f\searrow$ for $x\ge 2$. so by int test $\sum_{n=2}^{\infty}\frac{1}{n\ln n} \ge \int_2^{\infty}\frac{1}{x\ln x}dx = \int_{\ln 2}^{\infty}\frac{1}{u}du$ w/ $u=\ln x$, $du=dx$

$= \cancel{\ln|u|\Big|_{\ln 2}^{\infty}\to\infty} = \lim_{T\to\infty}\ln u\Big|_{\ln 2}^{T}\to\infty$ as $T\to\infty$. $\therefore$ the series diverges

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7.) $\sum_{n=1}^{\infty}\frac{n\sin^2 n}{1+n^3}$. 1st notice $\sin^2 n\le 1$ so $\frac{n\sin^2 n}{1+n^3}\le\frac{n}{1+n^3}$ and $n^3+1\ge n^3$

so $\frac{1}{n^3+1}\le\frac{1}{n^3} \Rightarrow \frac{n\sin^2 n}{1+n^3}\le\frac{n}{n^3}=\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{n\sin^2 n}{1+n^3}\le\sum_{n=1}^{\infty}\frac{1}{n^2}$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series w/ $p=2$, $\sum_{n=1}^{\infty}\frac{n\sin^2 n}{1+n^3}$ converges by comparison test

8.) $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ compare w/ $\sum_{n=1}^{\infty} \frac{1}{n}$ known to diverge. consider $\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}}$

study $\lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}}$ substitute $y=\frac{1}{x}$ so $\lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = \lim_{y\to 0^+} \frac{\sin y}{y} = \lim_{y\to 0^+} \cos y = 1$ by L'hôpital

so $\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1 > 0$ so by limit comparison $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ diverges

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9.) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$ is alternating just check if $\frac{\sqrt{n}}{2n+3} \searrow$ and $\frac{\sqrt{n}}{2n+3} \to 0$.

1st $\lim_{n\to\infty} \frac{\sqrt{n}}{2n+3} = \lim_{n\to\infty} \frac{1}{2\sqrt{n} + \frac{3}{\sqrt{n}}} = 0$. 2nd consider $f(x) = \frac{\sqrt{x}}{2x+3}$, then $f'(x) = \frac{\frac{1}{2\sqrt{x}}(2x+3) - 2\sqrt{x}}{(2x+3)^2}$

$= \frac{\frac{3}{2}x^{-\frac{1}{2}} - \sqrt{x}}{(2x+3)^2} < 0$ for large enough x so $\frac{\sqrt{n}}{2n+3} \searrow$. so by AST. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+3}$ converges.

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10) $\sum_{n=1}^{\infty} (-1)^n \tan^{-1}(n)$. is alternating just check if $\tan^{-1}(n) \searrow$ and $\tan^{-1}(n) \to 0$

but $\lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2} \neq 0$ so by Test for Divergence the series diverges.

11.) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ use Ratio test. then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$

$= \lim_{n\to\infty} (n+1) \frac{n^n}{(n+1)(n+1)^n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e}$

since $\frac{1}{e} < 1 \Rightarrow$ series converges abs.

12.) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ use root test. then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

since $e > 1 \Rightarrow$ series diverges.

13.) let $s_n = 3 - n2^{-n}$ be ~~nth~~ partial sum of $\sum_{n=1}^{\infty} a_n$, ~~if~~ then $a_n = s_{n+1} - s_n$

so $s_{n+1} = 3 - (n+1)2^{-(n+1)} \Rightarrow a_n = 3 - \frac{n+1}{2^{n+1}} - \left(3 - \frac{n}{2^n}\right) = \frac{n}{2^n} - \frac{n+1}{2^{n+1}} = \frac{2n}{2^{n+1}} - \frac{n+1}{2^{n+1}}$

$= \frac{n+1}{2^{n+1}}$. then $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3 - \frac{n}{2^n}\right) = 3 - \lim_{n\to\infty} \frac{n}{2^n}$

study $\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$ by L'hôpital. So $\lim_{n\to\infty} \left(3 - \frac{n}{2^n}\right) = 3$

$\Rightarrow \sum_{n=1}^{\infty} a_n = 3$