MA 2733 Practice Final Exam Solutions

1.) want $\text{comp}_{\vec a}\vec b = \text{comp}_{\vec b}\vec a \Rightarrow \frac{\vec a\cdot\vec b}{\|\vec a\|} = \frac{\vec b\cdot\vec a}{\|\vec b\|}$ either $\vec a\cdot\vec b=0$ or if $\vec a\cdot\vec b\neq 0 \Rightarrow \|\vec a\|=\|\vec b\|$

$\therefore$ $\vec a,\vec b$ must have same length or are $\perp$ to each other.
$\vec a,\vec b\neq 0$

2.) want $\text{Proj}_{\vec a}\vec b = \text{Proj}_{\vec b}\vec a \Rightarrow \frac{\vec a\cdot\vec b}{\|\vec a\|^2}\vec a = \frac{\vec b\cdot\vec a}{\|\vec b\|^2}\vec b$ if $\vec a\cdot\vec b=0$ then ok otherwise if $\vec a\cdot\vec b\neq 0$

$\Rightarrow \|\vec a\|^2\vec b = \|\vec b\|^2\vec a \Rightarrow \vec b = \frac{\|\vec b\|^2}{\|\vec a\|^2}\vec a$ either $\vec b = \frac{\|\vec b\|^2}{\|\vec a\|^2}\vec a$, $\vec a,\vec b\neq 0$ or are $\perp$ to each other

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3.) pts $A=(3,0,-1)$, $B=(-2,-2,3)$, $C=(7,1,-4)$ consider $\vec a = A-B = \langle 5,2,-4\rangle$, $\vec b = A-C = \langle -4,-1,3\rangle$

then $\vec n = \vec a\times\vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ 5 & 2 & -4 \\ -4 & -1 & 3 \end{vmatrix} = \vec i\begin{vmatrix} 2 & -4 \\ -1 & 3\end{vmatrix} - \vec j\begin{vmatrix} 5 & -4 \\ -4 & 3\end{vmatrix} + \vec k\begin{vmatrix} 5 & 2 \\ -4 & -1\end{vmatrix} = \langle 2,1,3\rangle$ so $\vec n\cdot((x,y,z)-A)=0$

so $\vec n\cdot\langle x-3, y-0, z+1\rangle = 0 \Rightarrow 2(x-3)+(y-0)+3(z+1)=0$ or $2x+y+3z=9$

4.) pt $A=(3,5,-1)$ line: $x=4-t$, $y=2t-1$, $z=-3t$, $t=0$: $B=(4,-1,0)$, $t=1$ $C=(3,1,-3)$

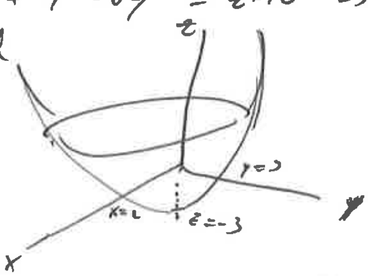
so $\vec a = A-B = \langle -1,6,-1\rangle$, $\vec b = A-C = \langle 0,4,2\rangle$ so $\vec n = \vec a\times\vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ -1 & 6 & -1 \\ 0 & 4 & 2 \end{vmatrix} = \vec i\begin{vmatrix} 6 & -1 \\ 4 & 2\end{vmatrix} - \vec j\begin{vmatrix} -1 & -1 \\ 0 & 2\end{vmatrix} + \vec k\begin{vmatrix} -1 & 6 \\ 0 & 4\end{vmatrix}$

so $\vec n = \langle 16,2,-4\rangle$. Then $\vec n\cdot\langle x-3, y-5, z+1\rangle = 0 \Rightarrow 16(x-3)+2(y-5)-4(z+1)=0$

$\Rightarrow 16x+2y-4z=-34$

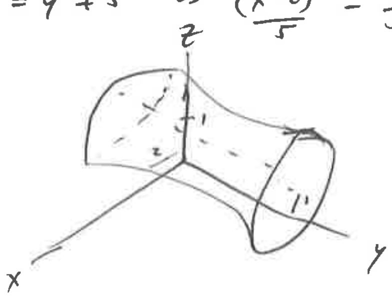
5.) $x^2+y^2-2x-6y-z+10=0 \Rightarrow x^2-2x + y^2-6y = z-10 \Rightarrow x^2-2x+4+y^2-6y+9 = z-10+4+9$

$\Rightarrow z+3=(x-2)^2+(y-3)^2$: elliptic paraboloid



6.) $x^2-y^2+z^2-4x-2z=0 \Rightarrow x^2-4x + z^2-2z = y^2 \Rightarrow x^2-4x+4+z^2-2z+1 = y^2+5$

$\Rightarrow (x-2)^2+(z-1)^2 = y^2+5 \Rightarrow \frac{(x-2)^2}{5} - \frac{y^2}{5} + \frac{(z-1)^2}{5} = 1$: hyperboloid of one sheet



7.) $\lim_{t\to 1} \left\langle \frac{t^2-t}{t-1}, \sqrt{t+8}, \frac{\sin(\pi t)}{\ln t} \right\rangle = \lim_{t\to 1} \left\langle \frac{t(t-1)}{t-1}, \sqrt{t+8}, \frac{\pi\cos(\pi t)}{\frac{1}{t}} \right\rangle \overset{\text{L'H in 3rd comp.}}{=} \lim_{t\to 1} \langle t, \sqrt{t+8}, \pi t\cos(\pi t) \rangle$

$= \langle 1, 3, -\pi \rangle$

8.) $\lim_{t\to\infty} \left\langle te^{-t}, \frac{t^3+t}{2t^3-1}, t\sin(\tfrac{1}{t}) \right\rangle = \lim_{t\to\infty} \left\langle \frac{t}{e^t}, \frac{1+\frac{1}{t^2}}{2-\frac{1}{t^3}}, \frac{\sin(\frac{1}{t})}{\frac{1}{t}} \right\rangle \overset{\text{L'H in 1st, 3rd comp.}}{=} \lim_{t\to\infty} \left\langle \frac{1}{e^t}, \frac{1+\frac{1}{t^2}}{2-\frac{1}{t^3}}, \cos(\tfrac{1}{t}) \right\rangle$

$= \langle 0, \frac{1}{2}, 1 \rangle$

9.) $\vec{r}(t) = \langle t^2-2t, 1+3t, \sin(2t) \rangle$, then $\vec{r}\,'(t) = \langle 2t-2, 3, 2\cos(2t) \rangle$ so $\vec{r}\,'(2) = \langle 2, 3, 2\cos 4 \rangle$

so $\|\vec{r}\,'(2)\| = \sqrt{4+9+4\cos^2 4} = \sqrt{13+4\cos^2(4)}$ so $\vec{T}(2) = \frac{1}{\sqrt{13+4\cos^2(4)}} \langle 2, 3, 2\cos(4) \rangle$

10.) $\vec{r}(t) = \langle \sin^2 t, \cos^2 t, \tan^2 t \rangle$, then $\vec{r}\,'(t) = \langle 2\sin t\cos t, -2\cos t\sin t, 2\tan t\sec^2 t \rangle$

so $\vec{r}\,'(\frac{\pi}{4}) = \langle \frac{2}{2}, -\frac{2}{2}, 2(\sqrt{2})^2 \rangle = \langle 1, -1, 4 \rangle$ and $\|\vec{r}\,'(\frac{\pi}{4})\| = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$

so $\vec{T}(\frac{\pi}{4}) = \frac{1}{3\sqrt{2}} \langle 1, -1, 4 \rangle$

11.) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ so $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$ so $\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t+\cos^2 t+1} = \sqrt{2}$

so $\vec{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$, thus $\vec{T}\,'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$, so $\|\vec{T}\,'(t)\| = \frac{\sqrt{\cos^2 t+\sin^2 t}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

so $\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$, then $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} =$

$= \vec{i} \begin{vmatrix} \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}} \\ -\sin t & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} -\frac{1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}} \\ -\cos t & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} -\frac{1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}}\cos t \\ -\cos t & -\sin t \end{vmatrix} = \langle \frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}} \rangle$

12.) $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$, so $\vec{r}\,'(t) = \langle 2t, 2t^2, 1 \rangle$ so $(1, \frac{2}{3}, 1) \Rightarrow t = 1$ so $\vec{r}\,'(1) = \langle 2, 2, 1 \rangle$

and $\|\vec{r}\,'(t)\| = \sqrt{4t^4+4t^2+1} = \sqrt{(2t^2+1)^2} = 2t^2+1$ so $\vec{T}(t) = \frac{1}{2t^2+1} \langle 2t, 2t^2, 1 \rangle$, then $\vec{T}(1) = \frac{1}{3}\langle 2, 2, 1 \rangle$

$\vec{T}\,'(t) = \left\langle \frac{2(2t^2+1) - 2t(4t)}{(2t^2+1)^2}, \frac{4t(2t^2+1) - 2t^2(4t)}{(2t^2+1)^2}, \frac{-4t}{(2t^2+1)^2} \right\rangle = \frac{1}{(2t^2+1)^2} \langle -4t^2+2, 4t^3+4t, -4t \rangle$

so $\vec{T}\,'(1) = \frac{1}{9} \langle -2, 8, -4 \rangle$ and $\|\vec{T}\,'(1)\| = \frac{1}{9}\sqrt{4+64+16} = \frac{1}{9}\sqrt{84} = \frac{2}{9}\sqrt{21}$ so $\vec{N}(1) = \frac{1}{2\sqrt{21}} \langle -2, 8, -4 \rangle$

so $\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{\sqrt{21}} & \frac{4}{\sqrt{21}} & \frac{-2}{\sqrt{21}} \end{vmatrix} = \vec{i} \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{4}{\sqrt{21}} & \frac{-2}{\sqrt{21}} \end{vmatrix} - \vec{j} \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{\sqrt{21}} & \frac{-2}{\sqrt{21}} \end{vmatrix} + \vec{k} \begin{vmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{-1}{\sqrt{21}} & \frac{4}{\sqrt{21}} \end{vmatrix} = \langle \frac{-8}{3\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{10}{3\sqrt{21}} \rangle$

13.) $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ so, $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$, $\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$

so $\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \vec{i} \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}$

$= \langle 6t^2, -6t, 2 \rangle$ and $\|\vec{r}\,'(t)\| = \sqrt{9t^4+4t^2+1}$ and $\|\vec{r}\,' \times \vec{r}\,''\| = \sqrt{36t^4+36t^2+4}$

so $\kappa(t) = \frac{\sqrt{36t^4+36t^2+4}}{(9t^4+4t^2+1)^{\frac{3}{2}}}$