MA 3053 Practice Exam 1

1.) pf:
Suppose $f(x)=f(y)$ for some $x,y\in X$. Then $f(f(x))=f(f(y)) \Rightarrow (f\circ f)(x)=(f\circ f)(y) \Rightarrow x=y$.
$\therefore f$ is 1-1. Now for any $y\in X$ consider $f(x)=y$. Then $f(f(x))=f(y) \Rightarrow (f\circ f)(x)=f(y)$
$\Rightarrow x=f(y)\in X$ as $y\in X$. $\therefore x\in X$. Thus f is ~~onto~~ onto and hence f is a bijection □

2.) pf:
Since f is decreasing, By def if $x_1<x_2 \Rightarrow f(x_1)>f(x_2)$ and if $x_1>x_2$ then $f(x_2)>f(x_1)$
Thus if $x_1\neq x_2$, there are two cases $x_1<x_2$ or $x_1>x_2$. If $x_1<x_2 \Rightarrow f(x_1)>f(x_2)$ and
if $x_2<x_1 \Rightarrow f(x_2)>f(x_1)$, both implying $f(x_1)\neq f(x_2)$. Thus f is 1-1. □

3.) pf:
(1) let $x\in f^{-1}\left(\bigcup_{\alpha\in A} P_\alpha\right) \Rightarrow f(x)\in \bigcup_{\alpha\in A} P_\alpha \Rightarrow f(x)\in P_\alpha$ for some $\alpha\in A$. $\Rightarrow x\in f^{-1}(P_\alpha)$ for some $\alpha\in A$
$\Rightarrow x\in \bigcup_{\alpha\in A} f^{-1}(P_\alpha)$. Since x arb. $\Rightarrow f^{-1}\left(\bigcup_{\alpha\in A} P_\alpha\right)\subseteq \bigcup_{\alpha\in A} f^{-1}(P_\alpha)$.
(2) let $x\in \bigcup_{\alpha\in A} f^{-1}(P_\alpha) \Rightarrow x\in f^{-1}(P_\alpha)$ for some $\alpha\in A \Rightarrow f(x)\in P_\alpha$ for some $\alpha\in A$. $\Rightarrow f(x)\in \bigcup_{\alpha\in A} P_\alpha$
$\Rightarrow x\in f^{-1}\left(\bigcup_{\alpha\in A} P_\alpha\right)$. Since x arb. $\Rightarrow f^{-1}\left(\bigcup_{\alpha\in A} P_\alpha\right)\supseteq \bigcup_{\alpha\in A} f^{-1}(P_\alpha)$. $\therefore$ by (1) and (2) $\Rightarrow f^{-1}\left(\bigcup_{\alpha\in A} P_\alpha\right)=\bigcup_{\alpha\in A} f^{-1}(P_\alpha)$ □

4.) pf:
(1) let $f(x)\in f\left(\bigcap_{\alpha\in A} P_\alpha\right) \Rightarrow x\in \bigcap_{\alpha\in A} P_\alpha \Rightarrow x\in P_\alpha$ for all $\alpha\in A$. $\Rightarrow f(x)\in f(P_\alpha)$ for all $\alpha\in A$. $\Rightarrow f(x)\in \bigcap_{\alpha\in A} f(P_\alpha)$.
since $f(x)$ was arb. $\Rightarrow f\left(\bigcap_{\alpha\in A} P_\alpha\right)\subseteq \bigcap_{\alpha\in A} f(P_\alpha)$.
(2) let $f(x)\in \bigcap_{\alpha\in A} f(P_\alpha) \Rightarrow f(x)\in f(P_\alpha)$ for all $\alpha\in A$. Since f is 1-1 $\Rightarrow x\in P_\alpha$ for all $\alpha\in A$. $\Rightarrow x\in \bigcap_{\alpha\in A} P_\alpha$
$\Rightarrow f(x)\in f\left(\bigcap_{\alpha\in A} P_\alpha\right)$. Since $f(x)$ was arb $\Rightarrow \bigcap_{\alpha\in A} f(P_\alpha)\subseteq f\left(\bigcap_{\alpha\in A} P_\alpha\right)$. □

5.) pf:
refl: does there exist $z\in\mathbb{N}^+$ s.t. $xz=x$? Yes pick $z=1$. antisym: let $x\preceq y$ and $y\preceq x$. $\Rightarrow$ there are $z_1,z_2\in\mathbb{N}^+$
s.t. $xz_1=y$ and $yz_2=x$. So $xz_1z_2=yz_2=x \Rightarrow z_1z_2=1 \Rightarrow z_1=z_2=1$. Trans: let $x\preceq y$ and $y\preceq z$
$\therefore$ there are $z_1,z_2\in\mathbb{N}^+$ s.t. $xz_1=y$ and $yz_2=z$. So $xz_1z_2=yz_2=z \Rightarrow x(z_1z_2)=z$.
Since $z_1,z_2\in\mathbb{N}^+ \Rightarrow z_1z_2\in\mathbb{N}^+$. $\therefore \preceq$ is a partial ordering □

6.) pf:
refl: Is it true $f\preceq f$? By def of $\preceq$ $f(x)\leq f(x)$ is always true so yes. antisym: let $f\preceq g$ and $g\preceq f$
$\therefore f(x)\leq g(x)$ and $g(x)\leq f(x)$ for all $x\in X$. $\Rightarrow f(x)=g(x)$ for all $x\in X$, thus $f=g$. Trans: let $f\preceq g$
and $g\preceq h$. $\therefore f(x)\leq g(x)$ and $g(x)\leq h(x)$ for all $x\in X$. $\Rightarrow f(x)\leq g(x)\leq h(x)$ for all $x\in X$
So $f(x)\leq h(x)$ for all $x\in X$ $\therefore f\preceq h$. $\therefore \preceq$ is a partial ordering □

7.) pf:

refl: is it true $x \sim x$? by def of $\sim$, consider $f(x)=f(x)$ which is always true as f is a func. $\therefore x \sim x$. sym: let $x \sim y$. $\Rightarrow f(x)=f(y) \Rightarrow f(y)=f(x)$ so $y \sim x$. Trans: let $x \sim y$ and $y \sim z$ $\therefore f(x)=f(y)$ and $f(y)=f(z) \Rightarrow f(x)=f(z)$ so $x \sim z$. $\therefore \sim$ is an equiv. relation.

notice $\sim$ is collecting the preimages together so $X/\sim = \{x \in X : f(x)=f(y) \text{ for some } y \in X\} =$

$= \{x \in X : f \text{ is 1-1}\}$. □

8.) pf:

refl: is it true $(a,b) \sim (a,b)$? notice $ab = ba$ as multi is comm. $\therefore (a,b) \sim (a,b)$.

sym: let $(a,b) \sim (c,d)$. $\Rightarrow ad = bc \Rightarrow bc = ad \Rightarrow cb = da$ as multi is comm. $\therefore (c,d) \sim (a,b)$.

trans: let $(a,b) \sim (c,d)$ and $(c,d) \sim (x,y)$. $\Rightarrow ad = bc$ and $cy = dx$. $\Rightarrow ady = bcy \Rightarrow ady = bdx$

$\Rightarrow ay = bx$ and as multi is comm. $\Rightarrow (a,b) \sim (x,y)$ so $\sim$ is an equiv. relation. □

9.) pf:

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