MA 3053 Practice Exam 2 Solutions

1.) Consider the following truth table:

p	q	$p\Rightarrow q$	$p\wedge(p\Rightarrow q)$	$(p\wedge(p\Rightarrow q))\Rightarrow q$
0	0	1	0	1
1	0	0	0	1
0	1	1	0	1
1	1	1	1	1

$\square$

2.) Consider the following truth table:

p	q	$p\Rightarrow q$	$\neg p$	$\neg p \vee q$
0	0	1	1	1
1	0	0	0	0
0	1	1	1	1
1	1	1	0	1

Can see the 3rd and 5th columns are the same $\square$

3.) Suppose $\sqrt{10}$ is rat'l. Then $\exists\, p,q\in\mathbb{Z}$, $q\neq 0$ s.t. $\sqrt{10}=\frac{p}{q}$ and p,q have no common factors. $\Rightarrow p^2=10q^2$ $\Rightarrow p^2\equiv_{10}0$, since mod arith. preserves multi. $\Rightarrow p\equiv_{10}0$ so $\exists\, t\in\mathbb{Z}$ s.t. $p=10t$. So $100t^2=10q^2$ $\Rightarrow q^2=10t^2 \Rightarrow q^2\equiv_{10}0$ so $q\equiv_{10}0$ so q has a factor of 10 contradiction $\square$

4.) Consider $\sqrt2$. Know $\sqrt2$ is irrat'l. Then $\sqrt2^{\sqrt2}$ is either irrat'l or rat'l. If $\sqrt2^{\sqrt2}$ is rat'l done. Otherwise $\sqrt2^{\sqrt2}$ is irrat'l. Then $\left(\sqrt2^{\sqrt2}\right)^{\sqrt2}=\sqrt2^{2}=2$. $\square$

5.) By MVT $\exists\, c\in(-6,6)$ s.t. $f(6)-f(-6)=f'(c)(6--6)$. $\Rightarrow f(6)--f(6)=f'(c)\,26$ since f is odd. So $2f(6)=26f'(c)$. Since $6>0 \Rightarrow f'(c)=\frac{f(6)}{6}$. $\square$

6.) Suppose that both $a<\sqrt c$ and $b<\sqrt c$. $\therefore ab<\sqrt c\cdot\sqrt c \Rightarrow ab<c$ contradicting the fact that $ab=c$. $\therefore$ must have one of $a\le\sqrt c$ or $b\le\sqrt c$. $\square$

7.) Base case: $n=0$: $LHS=\sum_{k=0}^{0}k^2=0^2=0$. $RHS=\frac{0(0+1)(2\cdot0+1)}{6}=0$. $\therefore LHS=RHS$. Inductive step: Suppose the statement is true for some $N\in\mathbb{N}$. Consider $\sum_{k=0}^{N+1}k^2=\sum_{k=0}^{N}k^2+(N+1)^2=\frac{N(N+1)(2N+1)}{6}+(N+1)^2$ by assump.

$\therefore \sum_{k=0}^{N+1}k^2=\frac{N+1}{6}\left(N(2N+1)+6(N+1)\right)=\frac{N+1}{6}\left(2N^2+N+6N+6\right)=\frac{N+1}{6}\left(2N^2+7N+6\right)=\frac{(N+1)(N+2)(2N+3)}{6}$

$\therefore$ by principle of induction, the result holds. $\square$

8.) Suppose $p(n)=an^3+bn^2+cn+d$. Then $p(0)=d$. OTOH $p(0)=\sum_{k=0}^{0}k^2=0$. $\therefore p(0)=0$. $p(1)=a+b+c$ and $p(1)=\sum_{k=0}^{1}k^2=0^2+1^2=1$ $\therefore a+b+c=1$. $p(2)=8a+4b+2c$ and $p(2)=\sum_{k=0}^{2}k^2=0^2+1^2+2^2=5$ so $8a+4b+2c=5$. Then $p(3)=27a+9b+3c$ and $p(3)=\sum_{k=0}^{3}k^2=0^2+1^2+2^2+3^2=14 \Rightarrow 27a+9b+3c=14$.

So have a system of eq.:
$$\begin{cases} a+b+c=1\\ 8a+4b+2c=5\\ 27a+9b+3c=14\end{cases}$$
Solving this yields $a=\frac13$, $b=\frac12$, $c=\frac16$.
To prove it using induction see #7. $\square$

9.) Let $a_1,\dots,a_n$ be rat'l #s. $\therefore \exists\, p_j,q_j\in\mathbb{Z}$, $q_j\neq0$ s.t. $a_j=\frac{p_j}{q_j}$ for $j=1,\dots,n$. Then $a_1a_2\cdots a_n=\frac{p_1}{q_1}\cdot\frac{p_2}{q_2}\cdots\frac{p_n}{q_n}=\frac{p_1\cdots p_n}{q_1\cdots q_n}$. But $p_1\cdots p_n\in\mathbb{Z}$ and $q_1\cdots q_n\in\mathbb{Z}$ and nonzero. $\therefore a_1a_2\cdots a_n$ is rat'l. $\sqrt2$ is irrat'l and since $\frac{1}{\sqrt2}=\frac{\sqrt2}{2}\Rightarrow\frac{1}{\sqrt2}$ irrat'l, but $\sqrt2\cdot\frac{1}{\sqrt2}=1$ $\square$