Name:_____

Math 3053 Section 01 *Practice Final Exam* *November 14, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Let $f : \mathbb{R} \to [0, 1]$ be a continuous function. Prove that $g(x) = \sin(f(x))$ is continuous on $\mathbb{R}$.

2. Prove that $f(x) = |x|$ is a continuous function on $\mathbb{R}$.

3. Let $p, q \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^p \sin\left(\dfrac{1}{x^q}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\,. \end{cases}$$

For which values of p and q is f continuous at $x = 0$? Be sure to justify.

4. Suppose f is a continuous function on $[a, b]$ and g is a continuous function on $[b, c]$ such that $f(b) = g(b)$. Define $h : [a, c] \to \mathbb{R}$ by

$$h(x) = \begin{cases} f(x) & \text{if } a \leq x \leq b \\ g(x) & \text{if } b \leq x \leq c\,. \end{cases}$$

Prove that h is continuous on $[a, c]$.

5. Let X be a countable set of real numbers and fix a to be a real number. Define the set

$$a + X = \{a + x : x \in X\}.$$

Prove that $a + X$ is countable.

6. Let X be a countable set of real numbers and fix a to be a nonzero real number. Define the set

$$\frac{X}{a} = \left\{\frac{x}{a} : x \in X\right\}.$$

Prove that $\dfrac{X}{a}$ is countable.

7. Let $d = \gcd(a, b)$ where $a, b \in \mathbb{N}$. If $a = da'$ and $b = db'$, show that $\gcd(a', b') = 1$.

8. Let $d = \gcd(a, b)$ where $a, b \in \mathbb{N}$. Prove that $\dfrac{a}{d}$ and $\dfrac{b}{d}$ are relatively prime.

9. We showed $\mathbb{R}$ is uncountable by proving $(0, 1)$ is uncountable. By assuming $\mathbb{R}$ is uncountable, prove that the interval $(0, 1)$ is uncountable by constructing a map from $(0, 1)$ to $\mathbb{R}$ and demonstrating the map is a bijection.