MA 3053 Practice Final Exam Solutions

1.) 1^{st} notice $\forall x,y \in \mathbb{R}$, $|\sin x - \sin y| = \left|\int_y^x \cos t\,dt\right| \le \left|\int_y^x dt\right| = |x-y|$ since $-1 \le \cos t \le 1$. $\therefore$ $|\sin x - \sin y| \le |x-y|$ $(*)$. Now since f is cont. on $\mathbb{R}$, $\forall \varepsilon > 0$ choose $\delta > 0$ s.t. if $|x-a|<\delta$ then $|f(x)-f(a)|<\varepsilon$. Consider $|\sin(f(x)) - \sin(f(a))| \le |f(x)-f(a)| < \varepsilon$ by $(*)$. $\therefore$ $\sin(f)$ is cont. at $x=a$. This works $\forall a \in \mathbb{R}$. $\therefore$ $\sin(f)$ is cont. on $\mathbb{R}$. $\square$

2.) $\forall \varepsilon > 0$ choose $\delta = \varepsilon$. If $|x-a|<\delta$. ~~then~~ Consider $\big||x|-|a|\big| \le |x-a| < \delta = \varepsilon$. by $\Delta \le$. $\therefore$ $f(x)=|x|$ is cont. at $x=a$. This works $\forall a \in \mathbb{R}$ $\therefore$ $f(x)=|x|$ is cont. on $\mathbb{R}$

3.) 1^{st} notice $-1 \le \sin\left(\frac{1}{x^q}\right) \le 1$ $\forall q \in \mathbb{R}\setminus\{0\}$. Secondly if $p<0$ then $\lim\limits_{x\to 0^+} x^p = \infty$ and $\lim\limits_{x\to 0^-} x^p = \pm\infty$. If $p=0$ $f(x) = \sin\left(\frac{1}{x^q}\right)$ $x\neq 0$ then f has no limit at $x=0$. Next if $p>1$ then $\lim\limits_{x\to 0} x^p = 0$ by limit laws so by squeeze thm $\lim\limits_{x\to 0} x^p \sin\left(\frac{1}{x^q}\right) = 0 = f(0)$. Finally if $0<p<1$ then for odd p, $\lim\limits_{x\to 0} x^p = 0$ use squeeze thm again. If p-even f is not defined for negative #s $\therefore$ study $\lim\limits_{x\to 0^+} x^p = 0$ so $\lim\limits_{x\to 0^+} x^p \sin\left(\frac{1}{x^q}\right) = 0 = f(0)$. $\therefore$ f is cont. on $\mathbb{R}$ for $q \in \mathbb{R}\setminus\{0\}$, $p>0$, and p odd when $0<p<1$ and cont. on $[0,\infty)$ for $q \in \mathbb{R}\setminus\{0\}$, $p>0$ and p-even when $0<p<1$. $\square$

4.) notice for $a \le x < b$, $h(x) = f(x)$ and f is cont. on $[a,b)$ so h is cont. on $a \le x < b$. If $b < x \le c$ then $h(x) = g(x)$ and g is cont. on $(b,c]$ so h is cont. on $b < x \le c$. The only pt in question is $x=b$. notice $\lim\limits_{x\to b^+} h(x) = \lim\limits_{x\to b^+} g(x) = g(b)$ by cont. of g. Similarly $\lim\limits_{x\to b^-} h(x) = \lim\limits_{x\to b^-} f(x) = f(b)$ by cont. of f. But know $f(b) = g(b)$ $\therefore$ $\lim\limits_{x\to b^+} h(x) = \lim\limits_{x\to b^-} h(x) = f(b) = g(b)$ so limit exists and can define $h(b) = f(b) = g(b)$. $\therefore$ h is cont at $x=b$ and thus on $[a,c]$ $\square$

5.) since X is countable, $\exists f: \mathbb{N} \to X$ bijection. Define $g: X \to a+X$ via $g(x) = x+a$. Given any $y \in a+X$ y has form $y = z+a$ for some $z \in X$. Consider $g(x) = y \Rightarrow x+a = z+a \Rightarrow x = z \in X$. so g is onto. next suppose $g(x) = g(y) \Rightarrow x+a = y+a \Rightarrow x = y$ so g is 1-1. $\therefore$ g is a bijection. now consider $g\circ f: \mathbb{N} \to a+X$. Then $g\circ f$ is a bijection and hence $a+X$ countable. $\square$

6.) since X countable, $\exists f: \mathbb{N} \to X$ bijection. Define $g: X \to \frac{X}{a}$ via $g(x) = \frac{x}{a}$. Given any $y \in \frac{X}{a}$ y has form $y = \frac{z}{a}$ for some $z \in X$. Consider $g(x) = y \Rightarrow \frac{x}{a} = \frac{z}{a} \Rightarrow x = z \in X$. $\therefore$ g is onto. now suppose $g(x) = g(y) \Rightarrow \frac{x}{a} = \frac{y}{a} \Rightarrow x = y$ so g is 1-1. $\therefore$ g is a bijection. now consider $g\circ f: \mathbb{N} \to \frac{X}{a}$. Thus $g\circ f$ is a bijection and hence $\frac{X}{a}$ countable $\square$

7.) Since $d = \gcd(a,b)$, $\exists m,n \in \mathbb{Z}$ s.t. $d = ma + nb \overset{\text{by assumption}}{=} mda' + ndb'$. $\Rightarrow 1 = ma' + nb'$. $\therefore$ $\gcd(a',b') = 1$. $\square$

8.) Since $d = \gcd(a,b)$, $\exists m,n \in \mathbb{Z}$ s.t. $d = ma + nb$. since $d \ne 0 \Rightarrow 1 = m\left(\frac{a}{d}\right) + n\left(\frac{b}{d}\right) \Rightarrow \gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$. $\square$

9.) Consider $f: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ via $f(x) = \tan^{-1} x$. Then f is a bijection. Define $g: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (0,1)$, via $g(x) = \frac{1}{\pi}x + \frac{1}{2}$. Clearly g is a bijection. now $g\circ f: \mathbb{R} \to (0,1)$, is a bijection. $\therefore$ since $\mathbb{R}$ is uncountable $\Rightarrow$ $(0,1)$ uncountable. $\square$