Practice Exam 2 Solutions

1.) Let P, Q be statements. Prove the de Morgans Laws.

pf:

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg P \wedge \neg Q$	$\neg(P \wedge Q)$	$\neg(P \vee Q)$
0	0	0	0	1	1	1	1	1	1
1	0	0	1	0	1	1	0	1	0
0	1	0	1	1	0	1	0	1	0
1	1	1	1	0	0	0	0	0	0
						(a)	(b)	(a)	(b)

□

2.) Let P, Q be statements. Prove $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

pf:

P	Q	$\neg P$	$\neg P \vee Q$	$P \Rightarrow Q$
0	0	1	1	1
1	0	0	0	0
0	1	1	1	1
1	1	0	1	1

□

3.) Prove $\sqrt{10}$ is irrational.

pf: s'pose $\sqrt{10}$ is rational. Then $\exists\, p, q \in \mathbb{Z}$ s.t. $\sqrt{10} = \frac{p}{q}$, $q \neq 0$ and p, q have no common terms.
$\Rightarrow 10q^2 = p^2 \Rightarrow p^2 \equiv_{10} 0 \Rightarrow p \equiv_{10} 0 \Rightarrow p = 10r$ for some $r \in \mathbb{Z}$. $\Rightarrow 10q^2 = 100r^2$
$\Rightarrow q^2 = 10r^2 \Rightarrow q^2 \equiv_{10} 0 \Rightarrow q \equiv_{10} 0 \Rightarrow \exists\, s \in \mathbb{Z}$ s.t. $q = 10s \Rightarrow p, q$ have a common factor. contradiction. □.

4.) Prove $\exists\, x, y$ irrational s.t. x^y is rational.

pf: consider $\sqrt{2}$. know $\sqrt{2}$ is irrational. Then $\sqrt{2}^{\sqrt{2}}$ is either irrational or rational. If $\sqrt{2}^{\sqrt{2}}$ is rational, done. otherwise if $\sqrt{2}^{\sqrt{2}}$ is irrational consider $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$
so $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ is rational. □.

5.) Prove $\forall\, n \in \mathbb{N}$. $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

pf: 1st base case: $n=0$: Then $LHS = (x+y)^0 = 1$, $RHS = \sum_{k=0}^{0} \binom{0}{k} x^{-k} y^k = 1$ so $LHS = RHS$.
Inductive step: s'pose statement is true for n. consider $(x+y)^{n+1} = (x+y)(x+y)^n =$
$= (x+y) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$ via assumption.
reindex in 2nd sum by $j = k+1 \Rightarrow (x+y)^{n+1} = \sum_{j=0}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=1}^{n+1} \binom{n}{j-1} x^{n-(j-1)} y^j =$
$= \sum_{j=0}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=1}^{n} \binom{n}{j-1} x^{n+1-j} y^j = \binom{n}{0} x^{n+1} + \sum_{j=1}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=1}^{n} \binom{n}{j-1} x^{n+1-j} y^j + \binom{n}{n} y^{n+1}$
$= \binom{n}{0} x^{n+1} + \sum_{j=0}^{n} \left[\binom{n}{j} + \binom{n}{j-1} \right] x^{n+1-j} y^j + \binom{n+1}{n+1} y^{n+1} = \binom{n+1}{0} x^{n+1} + \sum_{j=1}^{n} \binom{n+1}{j} x^{n+1-j} y^j + \binom{n+1}{n+1} y^{n+1}$
$= \sum_{j=0}^{n+1} \binom{n+1}{j} x^{n+1-j} y^j$ which shows $n+1$ case. ∴ via induction true $\forall\, n \in \mathbb{N}$. □

6.) Prove $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall n \in \mathbb{N}$.

pf:

1st base case: $n=0$ $LHS = \sum_{k=0}^{0} k^2 = 0^2 = 0$ and $RHS = \frac{0(1)(1)}{6} = 0$

Inductive step: s'pose true for n then consider $\sum_{k=0}^{n+1} k^2 = (n+1)^2 + \sum_{k=0}^{n} k^2 = (n+1)^2 + \frac{n(n+1)(2n+1)}{6} =$

$= n^2+2n+1 + \frac{n(n+1)(2n+1)}{6} = \frac{n+1}{6}(6n+6+2n^2+n) = \frac{n+1}{6}(n+2)(2n+3) = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

So this shows $n+1$ case. $\therefore$ by inductive hyp true $\forall n \in \mathbb{N}$. $\square$

7.) Prove $2^n > n$ $\forall n \in \mathbb{N}$

pf:

1st base case: $n=0$ $LHS = 2^0 = 1$ and $RHS = 0 \Rightarrow 1 > 0$ so true.

Inductive step: s'pose true for n then $2^{n+1} = 2 \cdot 2^n > 2n > n+1$ if $n \geq 1$

$\Rightarrow 2^{n+1} > n+1$ so true for $n+1$ case. $\therefore$ true $\forall n \in \mathbb{N}$ by inductive hyp. $\square$

8.) Assume $\exists$ poly $p(n)$, deg $p = \cancel{2}\ 3$ s.t. $p(n) = \sum_{k=0}^{n} k^2$. Find formula of $p(n)$.

pf:

S'pose $p(n) = an^3+bn^2+cn+d$. Then if $p(0) = d$ and $p(0) = \sum_{k=0}^{0} k^2 = 0 \Rightarrow d=0$

$p(1) = a+b+c$ and $\sum_{k=0}^{1} k^2 = 1 \Rightarrow a+b+c=1$ and $p(2) = 8a+4b+2c$, $p(2) = \sum_{k=0}^{2} k^2 = 1+4 = 5$

$\Rightarrow 8a+4b+2c = 5$ and $p(3) = \sum_{k=0}^{3} k^2 = 1+4+9 = 14$ and $p(3) = 27a+9b+3c$

So get system: $\begin{cases} a+b+c=1 \\ 8a+4b+2c=5 \\ 27a+9b+3c=14 \end{cases}$ solving this $\Rightarrow a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6} \Rightarrow p(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ $\square$

9.) Prove $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$ is always true

pf:

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$\{P \wedge (P \Rightarrow Q)\} \Rightarrow Q$
0	0	1	0	1
1	0	0	0	1
0	1	1	0	1
1	1	1	1	1

$\square$