MA 3053 Practice Final Exam

1.) Prove that $(f+g)'(a) = f'(a) + g'(a)$. provided $f'(a), g'(a)$ exists

pf:

Suppose $f'(a), g'(a)$ exist. consider $(f+g)'(a) = \lim_{x\to a} \frac{(f+g)(x) - (f+g)(a)}{x-a} = \lim_{x\to a} \frac{f(x)+g(x) - f(a) - g(a)}{x-a}$
$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} + \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(a) + g'(a)$. by limit laws. $\square$

2.) let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Prove $\lim_{x\to a} f(x)g(x) = LM$.

pf:

since limits of f, g exist at $x=a$ know $\forall \varepsilon > 0$ $\exists \delta_1 > 0$ s.t. if $|x-a| < \delta_1$ then $|f(x) - f(a)| < \frac{\varepsilon}{2(|M|+1)}$ and $\forall \varepsilon > 0$, $\exists \delta_2 > 0$ s.t. if $|x-a| < \delta_2$ then $|g(x) - M| < \frac{\varepsilon}{2(|C|+1)}$
where $|f(x)| < C$ for $|x-a| < \delta_1$. Pick $\delta = \min\{\delta_1, \delta_2\}$ and let $\varepsilon > 0$. if $|x-a| < \delta$
consider $|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| = |f(x)(g(x)-M) + M(f(x)-L)|$
$\leq |f(x)||g(x)-M| + |M||f(x)-L| \leq |C||g(x)-M| + |M||f(x)-L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\square$.

3.) Prove $aX = \{ax : x \in X\}$ is countable. if X is.

pf:

Since X is countable, $\exists f: \mathbb{N} \to X$ bijection. consider $g: X \to aX$ via $g(x) = ax$
then given any $ax \in aX$ $g(x) = ax \Rightarrow x \in X$ so g is onto. if $g(x) = g(y) \Rightarrow ax = ay$
$\Rightarrow x = y$ since $a \neq 0$. so g is bijection. define $h: \mathbb{N} \to aX$ via
$h = g^{-1} \circ f$, since g^{-1} bij. and composition of bij are bijection $\Rightarrow h$ bijection. so aX countable.

4.) Prove $X + a = \{x + a : x \in X\}$ is countable if X is.

pf:

since X is countable, $\exists f: \mathbb{N} \to X$ bij. consider $g: X \to a+X$ via $g(x) = x+a$
given any $x+a \in X+a$ $g(x) = x+a \Rightarrow x \in X$ so g is onto. if $g(x) = g(y) \Rightarrow a+x = a+y \Rightarrow x=y$.
So g is bij. define. $h: \mathbb{N} \to X+a$ via $h = g^{-1} \circ f$. like in (3) h is bij. so $X+a$ countable $\square$

5.) let $X \supseteq Y$ and X countable. Prove Y countable.

pf:

Since X countable $\exists f: \mathbb{N} \to X$ s.t. f is bij. define $g = f|_Y$ $g: \mathbb{N} \to Y$ then if $g(x) = g(y)$
$\Rightarrow f(x)|_Y = f(y)|_Y \Rightarrow x = y$ as f is 1-1. given any $y \in Y$ then $g(x) = y \Rightarrow f(x)|_Y = y \Rightarrow x \in \mathbb{N}$
since f is onto $\Rightarrow g$ is bijective. so Y countable. $\square$

6.) let $Y \subseteq X$ and Y uncountable. Prove X is uncountable.

pf: S'pose X is countable. then by (5) $\Rightarrow$ Y is countable. contradiction $\square$

7.) let $d = \gcd(a,b)$ and $a = da'$, $b = db'$. Prove $\gcd(a', b') = 1$.

pf: since $d = \gcd(a,b) \Rightarrow \exists\, m,n \in \mathbb{Z}$ s.t. $d = ma + nb \Rightarrow d = mda' + ndb'$
$\Rightarrow 1 = ma' + nb' \Rightarrow a', b'$ are relatively prime $\Rightarrow \gcd(a', b') = 1$. $\square$

8.) let $d = \gcd(a,b)$. Prove $\frac{a}{d}, \frac{b}{d}$ are relatively prime.

pf: since $d = \gcd(a,b)$, $\Rightarrow \exists\, m,n \in \mathbb{Z}$ s.t. $d = ma + nb \Rightarrow 1 = m(\frac{a}{d}) + n(\frac{b}{d})$
$\Rightarrow \frac{a}{d}, \frac{b}{d}$ are relatively prime. $\square$

9.) Assume $\mathbb{R}$ uncountable. Prove $(0,1)$ is uncountable.

pf: consider $f_1(x) = \tan x$, $f_1 : (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ on this interval seen f_1 is 1-1 and onto so a bijection. next consider $f_2(x) = \frac{\pi}{2}x$ then $f_2 : \mathbb{R} \to \mathbb{R}$ then clear f_2 is bijection. finally $f_3(x) = x+1$, $f_3 : \mathbb{R} \to \mathbb{R}$ is also clear to be a bijection.
Consider $h : (0,1) \to \mathbb{R}$ via $h = f_3 \circ f_1 \circ f_2$. compositions of bijections are still bijections.
so $|\mathbb{R}| = |(0,1)| \Rightarrow (0,1)$ uncountable. $\square$