MA 3053: Practice Exam 1 Solutions

1.) pf: S'pose $f(x_1)=f(x_2)$ for some $x_1, x_2 \in X$. Consider $g(f(x_1)) = g(f(x_2)) \Rightarrow (g\circ f)(x_1) = (g\circ f)(x_2)$.
Since $g\circ f$ is an injection $\Rightarrow x_1 = x_2$. $\therefore f$ is an injection $\square$.

2.) pf:
Consider $g(w_1)$ and $h(w_2)$ for $w_1, w_2 \in W$. Since $g, h: W \to X \Rightarrow g(w_1), h(w_2) \in X$. $\therefore$ set $x_1 = g(w_1)$ and $x_2 = h(w_2)$. S'pose $f(x_1) = f(x_2) \Rightarrow f(g(w_1)) = f(h(w_2)) \Rightarrow (f\circ g)(w_1) = (f\circ h)(w_2)$
By the assumption $f\circ g = f\circ h \Rightarrow g = h$ so $\therefore$ $g(w_1) = h(w_2)$ ie $x_1 = x_2$. $\therefore f$ is an injection $\square$

3.) pf:
(a) let $x \in f^{-1}\left(\bigcap_{\alpha\in A} P_\alpha\right) \Rightarrow f(x) \in \bigcap_{\alpha\in A} P_\alpha \Rightarrow f(x) \in P_\alpha$ for all $\alpha \in A$. $\Rightarrow x \in f^{-1}(P_\alpha)$ for all $\alpha\in A$.
$\Rightarrow x \in \bigcap_{\alpha\in A} f^{-1}(P_\alpha)$ $\therefore$ $f^{-1}\left(\bigcap_{\alpha\in A} P_\alpha\right) \subseteq \bigcap_{\alpha\in A} f^{-1}(P_\alpha)$.
(b) let $x \in \bigcap_{\alpha\in A} f^{-1}(P_\alpha) \Rightarrow x \in f^{-1}(P_\alpha)$ for all $\alpha\in A \Rightarrow f(x) \in P_\alpha$ for all $\alpha \in A$. $\Rightarrow f(x) \in \bigcap_{\alpha\in A} P_\alpha$
$\Rightarrow x \in f^{-1}\left(\bigcap_{\alpha\in A} P_\alpha\right)$. $\therefore \bigcap_{\alpha\in A} f^{-1}(P_\alpha) \subseteq f^{-1}\left(\bigcap_{\alpha\in A} P_\alpha\right)$.
$\therefore$ by (a) and (b) $\Rightarrow f^{-1}\left(\bigcap_{\alpha\in A} P_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(P_\alpha)$. $\square$

4.) (a) let $f(x) \in f\left(\bigcup_{\alpha\in A} P_\alpha\right) \Rightarrow x \in \bigcup_{\alpha\in A} P_\alpha \Rightarrow x \in P_\alpha$ for some $\alpha\in A$. $\Rightarrow f(x) \in f(P_\alpha)$ for some $\alpha \in A$.
$\therefore f(x) \in \bigcup_{\alpha\in A} f(P_\alpha)$. $\therefore f\left(\bigcup_{\alpha\in A} P_\alpha\right) \subseteq \bigcup_{\alpha\in A} f(P_\alpha)$.
(b) let $f(x) \in \bigcup_{\alpha\in A} f(P_\alpha) \Rightarrow f(x) \in f(P_\alpha)$ for some $\alpha\in A$. $\Rightarrow x \in P_\alpha$ for some $\alpha\in A$. $\Rightarrow x \in \bigcup_{\alpha\in A} P_\alpha$
$\Rightarrow f(x) \in f\left(\bigcup_{\alpha\in A} P_\alpha\right)$. $\therefore \bigcup_{\alpha\in A} f(P_\alpha) \subseteq f\left(\bigcup_{\alpha\in A} P_\alpha\right)$.
$\therefore$ by (a) and (b) $\Rightarrow f\left(\bigcup_{\alpha\in A} P_\alpha\right) = \bigcup_{\alpha\in A} f(P_\alpha)$. $\square$

5.) pf:
Check if $(a,b) \sim (a,b)$? This says $a+b = b+a$ but add. in $\mathbb{Z}$ is commutative $\therefore (a,b) \sim (a,b)$. so refl.
Let $(a,b) \sim (c,d) \Rightarrow a+d = b+c \Rightarrow b+c = a+d \Rightarrow c+b = d+a \Rightarrow (c,d) \sim (a,b)$. so sym.
Let $(a,b) \sim (c,d)$, $(c,d) \sim (x,y)$. $\Rightarrow a+d = b+c$ and $c+y = d+x$ $\Rightarrow$ $b+c+y = b+d+x$ ~~$c+b+d = x+y+b$~~
$\Rightarrow a+d+y = b+d+x \Rightarrow a+y = b+x \Rightarrow (a,b) \sim (x,y)$ so transitive $\square$.

6.) pf:
check if $(a,b) \sim (a,b)$? means $ab = ba$, but mult. in $\mathbb{Z}$ is commutative $\therefore (a,b) \sim (a,b)$. so ref.
let $(a,b) \sim (c,d) \Rightarrow ad = bc \Rightarrow bc = ad \Rightarrow cb = da \Rightarrow (c,d) \sim (a,b)$ $\therefore$ sym.
Finally let $(a,b) \sim (c,d)$ and $(c,d) \sim (x,y) \Rightarrow ad = bc$ and $cy = dx \Rightarrow bcy = bdx$
$\Rightarrow ady = bdx \Rightarrow ay = bx \Rightarrow (a,b) \sim (x,y)$ so transitive. $\square$

7.) pf:

check if $X \preceq X$? means $X \subseteq X$ true by def. $\therefore X \preceq X$ so ref. let $X \preceq Y$ & $Y \preceq X$ $\Rightarrow X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$ by def. $\therefore$ antisym. let $X \preceq Y$ and $Y \preceq Z$ $\Rightarrow X \subseteq Y$ and $Y \subseteq Z$. $\therefore$ if $x \in X$ then $x \in Y$ but $\Rightarrow x \in Z$. $\therefore X \subseteq Z$. so transitive. $\square$

8.) pf:

check if $f \preceq f$? means $f(x) \leq f(x)$ for all $x \in X$ which is always true $\therefore f \preceq f$ so ref. let $f \preceq g$ & $g \preceq f$ $\Rightarrow f(x) \leq g(x)$ and $g(x) \leq f(x)$ for all $x \in X$. $\therefore$ by def. of fncs. we have $f(x) = g(x)$ so $f = g$ $\therefore$ antisym. Finally let $f \preceq g$, $g \preceq h$. $\therefore f(x) \leq g(x)$ and $g(x) \leq h(x)$ for all $x \in X$. $\therefore f(x) \leq g(x) \leq h(x)$ for all $x \in X \Rightarrow f(x) \leq h(x)$ for all $x \in X$. $\therefore f \preceq h$ so transitive. $\square$

9.) pf:

Recall in $\mathbb{Z}/6\mathbb{Z}$, the equiv classes are $[0] = \{\ldots, -6, 0, 6, 12, \ldots\}$, $[1] = \{\ldots, -5, 1, 7, \ldots\}$ etc.
so

+	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	[3]	[4]	[5]	[0]
[2]	[2]	[3]	[4]	[5]	[0]	[1]
[3]	[3]	[4]	[5]	[0]	[1]	[2]
[4]	[4]	[5]	[0]	[1]	[2]	[3]
[5]	[5]	[0]	[1]	[2]	[3]	[4]

$\cdot$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[0]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]