MA 3053 Practice Exam 2 Solutions

1.) Consider the following truth table:

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$
0	0	1	0	1
1	0	0	0	1
0	1	1	0	1
1	1	1	1	1

$\square$

2.) Consider the following truth table:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
0	0	1	1	1
1	0	0	0	0
0	1	1	1	1
1	1	1	0	1

Can see these are prop. equivalent. $\square$

3.) Suppose $\sqrt{10}$ is rat'l. Then $\exists\, p, q \in \mathbb{Z}$, $q \neq 0$ s.t. $\sqrt{10} = \frac{p}{q}$ and p, q have no common factor.
$\Rightarrow p^2 = 10q^2 \Rightarrow p^2 \equiv_{10} 0 \Rightarrow p \equiv_{10} 0 \Rightarrow p = 10s$ for some $s \in \mathbb{Z}$. $\Rightarrow 100s^2 = 10q^2$
$\Rightarrow q^2 = 10s^2 \Rightarrow q^2 \equiv_{10} 0 \Rightarrow q \equiv_{10} 0$ $\therefore$ p, q have a common factor. Contradiction $\square$

4.) Consider $\sqrt{2}$, know $\sqrt{2}$ is irrat'l. Then $\sqrt{2}^{\sqrt{2}}$ is either rat'l or irrat'l. If $\sqrt{2}^{\sqrt{2}}$ is rat'l, done, otherwise $\sqrt{2}^{\sqrt{2}}$ is irrat'l. Then consider $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$. $\square$

5.) Base case: $n=1$ LHS $= (1+x)^1 = 1+x$, RHS $= 1+x$ $\therefore$ LHS $\geq$ RHS. Inductive step: Suppose the inequality is true for n. Consider $(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx)$ by assumption.
So $(1+x)^{n+1} \geq 1 + x + nx + nx^2 = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x$ since $nx^2 \geq 0$. $\therefore$ by ~~induction~~ principle of induction the result follows. $\square$

6.) Base cases: $n=0$ $(fg)^{(0)}(x) = (fg)(x)$. $n=1$. LHS $= (fg)'(x) = f'(x)g(x) + f(x)g'(x)$. By prod. rule.
RHS $= \sum_{k=0}^{1} \binom{1}{k} f^{(1-k)}(x) g^{(k)}(x) = f'(x)g(x) + f(x)g'(x)$. Inductive step: Suppose the equality is true for n,
consider $(fg)^{(n+1)}(x) = \left((fg)^{(n)}(x)\right)' = \left(\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)\right)'$ by assumption.

So $(fg)^{(n+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left[f^{(n-k)}(x) g^{(k)}(x)\right]' = \sum_{k=0}^{n} \binom{n}{k} \left(f^{(n-k+1)}(x) g^{(k)}(x) + f^{(n-k)}(x) g^{(k+1)}(x)\right)$

$= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)}(x) g^{(k)}(x) + \sum_{k=0}^{n} f^{(n-k)}(x) g^{(k+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)}(x) g^{(k)}(x) + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(n+1-k)}(x) g^{(k)}(x) + f^{(n+1)}(x) g(x)$

$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)}(x) g^{(k)}(x)$. $\therefore$ By principle of induction the result holds. $\square$

7.) Base case: $n=0$, LHS $= \sum_{k=0}^{0} k^2 = 0^2 = 0$, RHS $= \frac{0(0+1)(2\cdot 0+1)}{6} = 0$. Inductive step: Suppose the equality is true for n. Consider $\sum_{k=0}^{n+1} k^2 = \sum_{k=0}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$ by assumption.
So $\sum_{k=0}^{n+1} k^2 = (n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right) = \frac{n+1}{6}\left(2n^2 + n + 6n + 6\right) = \frac{n+1}{6}(2n^2 + 7n + 6) = \frac{(n+1)(n+2)(2n+3)}{6}$

$\therefore$ By principle of induction the result holds. $\square$

8.) Suppose $p(n) = an^3 + bn^2 + cn + d$. Notice $p(0) = d$. OTOH $p(0) = \sum_{k=0}^{0} k^2 = 0^2 = 0$. So $d = 0$. $p(1) = a + b + c$ and $p(1) = \sum_{k=0}^{1} k^2 = 0 + 1 = 1$ So $a + b + c = 1$. $p(2) = 8a + 4b + 2c$, OTOH $p(2) = \sum_{k=0}^{2} k^2 = 0 + 1 + 4 = 5$ so $8a + 4b + 2c = 5$. Finally $p(3) = 27a + 9b + 3c$. $p(3) = \sum_{k=0}^{3} k^2 = 0 + 1 + 4 + 9 = 14$

$\therefore$ have the system: $\begin{cases} a + b + c = 1 \\ 8a + 4b + 2c = 5 \\ 27a + 9b + 3c = 14 \end{cases}$ Solving this yields: $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$

$\therefore$ $p(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n+1)(2n+1)}{6}$

So ~~to~~ prove via induction see # 7. $\square$

9.) Let $a_1, \ldots, a_n$ be rat'l #'s. $\therefore \exists\, p_j, q_j \in \mathbb{Z}$, $q_j \neq 0$ for $j = 1, \ldots, n$ s.t. $a_j = \frac{p_j}{q_j}$ for $j = 1, \ldots, n$.
Then $a_1 a_2 \cdots a_n = \frac{p_1}{q_1} \cdots \cdots \frac{p_n}{q_n} = \frac{p_1 p_2 \cdots p_n}{q_1 \cdots q_n}$. but $p_1 \cdots p_n \in \mathbb{Z}$ and $q_1 \cdots q_n \in \mathbb{Z}$ and $q_1 \cdots q_n \neq 0$.

$\therefore$ $a_1 \cdots a_n$ is rat'l. Know $\sqrt{2}$, is irrat'l and $\frac{1}{\sqrt{2}}$ is irrat'l since $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. but $\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1$ not irrat'l $\square$