Name:______________________________

Math 3053 Section 01 | *Practice Final Exam* | *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Let f and g be real functions. Recall f is said to be differentiable at $x = a$ if the following limit exists:

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Suppose that f and g are both differentiable at $x = a$. Prove that $f + g$ is differentiable at $x = a$ and show that

$$(f + g)'(a) = f'(a) + g'(a)$$

2. Let f and g be real functions such that

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} g(x) = M$$

Prove that

$$\lim_{x \to a} f(x)g(x) = LM$$

3. Let X be a countable set of real numbers and fix a to be a nonzero real number. Define the set

$$aX = \{ax : x \in X\}.$$

Prove that aX is countable.

4. Let X be a countable set of real numbers and fix a to be a nonzero real number. Define the set

$$X + a = \{x + a : x \in X\}.$$

Prove that $X + a$ is countable.

5. Let X and Y be sets. Suppose $Y \subseteq X$ and X is countable. Prove that Y is countable.

6. Let X and Y be sets. Suppose $Y \subseteq X$ and Y is uncountable. Prove that X is uncountable.

7. Let $d = \gcd(a, b)$ where $a, b \in \mathbb{N}$. If $a = da'$ and $b = db'$, show that $\gcd(a', b') = 1$.

8. Let $d = \gcd(a, b)$ where $a, b \in \mathbb{N}$. Prove that $\frac{a}{d}$ and $\frac{b}{d}$ are relatively prime.

9. We showed $\mathbb{R}$ is uncountable by proving $(0, 1)$ is uncountable. By assuming $\mathbb{R}$ is uncountable, prove that the interval $(0, 1)$ is uncountable by constructing a map from $(0, 1)$ to $\mathbb{R}$ and demonstrating the map is a bijection.