MA 3053 Practice Final Solutions

1.) pf: Suppose $f'(a), g'(a)$ exist. Then $(f+g)'(a) = \lim_{x\to a} \frac{(f+g)(x) - (f+g)(a)}{x-a} = \lim_{x\to a} \frac{f(x)+g(x) - f(a) - g(a)}{x-a}$
$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} + \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(a)+g'(a)$. □

2.) pf: Since limits of f and g exist at a, know $\forall \varepsilon>0$, $\exists \delta_1>0$ s.t. if $0<|x-a|<\delta_1$ then $|f(x)-L| < \frac{\varepsilon}{2(|M|+1)}$ and $\exists \delta_2>0$ s.t. if $0<|x-a|<\delta_2$ then $|g(x)-M| < \frac{\varepsilon}{2(|L|+1)}$. Pick $\delta = \min\{\delta_1,\delta_2\}$. Then if $|x-a|<\delta$, then $|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| \le |f(x)||g(x)-M| + |M||f(x)-L| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ □

3.) pf: Since X is countable, $\exists f: \mathbb{N}\to X$ bijection. Define $g: X\to aX$ by $g(x) = ax$. Then given any $y\in aX$ y has the form $y = az$ for some $z\in X$. Then $g(x) = y \Rightarrow ax = az \Rightarrow x = z\in X$ as $a\neq 0$, so g onto.
Now suppose $g(x)=g(y) \Rightarrow ax = ay \Rightarrow x=y$ as $a\neq 0$. $\therefore$ g is 1-1. So g is a bijection.
Thus define $h:\mathbb{N}\to aX$ via $h = g\circ f$ and so h is a bijection hence aX countable. □

4.) pf: Since X is countable, $\exists f:\mathbb{N}\to X$ bijection. Define $g: X\to a+X$ by $g(x) = x+a$. Given any $y\in X+a$ y has form $y = z+a$ for some $z\in X$. Then $g(x) = y \Rightarrow a+x = a+z \Rightarrow x = z\in X$, so g onto.
Since $g(x)=g(y) \Rightarrow a+x = a+y \Rightarrow x=y$ so g is 1-1. So g is bijection. Define $h:\mathbb{N}\to a+X$ via $h = g\circ f$, thus h is a bijection so $X+a$ is countable □

5.) pf: Since X is countable, $\exists f:\mathbb{N}\to X$ bijection. Define $g:\mathbb{N}\to Y$ via $g = f|_Y$. If $g(x)=g(y)$ $\Rightarrow f(x)|_Y = f(y)|_Y$ and since f is 1-1 $\Rightarrow x=y$. So g 1-1. Next given any $y\in Y$ consider $g(x)=y$ $\Rightarrow f(x)|_Y = y$, but f is onto $\therefore x\in\mathbb{N}$. So g onto. $\therefore$ g bijection and hence Y is countable.

6.) pf: Suppose X is countable. Then by (5) above, Y is countable, contradiction. $\therefore$ X is uncountable. □

7.) pf: Since $d = \gcd(a,b)$, $\exists m,n\in\mathbb{Z}$ s.t. $d = ma+nb = mda'+ndb'$ by assumption. $\Rightarrow 1 = a'm + nb'$
$\therefore \gcd(a',b') = 1$.

8.) pf: Since $\gcd(a,b) = d \Rightarrow \exists m,n\in\mathbb{Z}$ s.t. $d = ma+nb \Rightarrow 1 = m\left(\frac{a}{d}\right) + n\left(\frac{b}{d}\right)$ as $d\neq 0$. $\Rightarrow \gcd\left(\frac{a}{d},\frac{b}{d}\right) = 1$. □

9.) pf: 1st consider $f:\mathbb{R}\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ via $f(x) = \tan^{-1}x$. Saw f is a bijection on this domain w/ codomain.
Next consider $g:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to(0,1)$ via $g(x) = ax+b$ for some $a,b\in\mathbb{R}$. Then consider $g\left(-\frac{\pi}{2}\right) = b - a\frac{\pi}{2}$, $g\left(\frac{\pi}{2}\right) = b + a\frac{\pi}{2}$
want $\begin{cases} b - a\frac{\pi}{2} = 0 \\ b + a\frac{\pi}{2} = 1\end{cases} \Rightarrow a = \frac{1}{\pi}$ and $b = 1$. Clearly g is a bijection from $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ to $(0,1)$.
So define $h(x) = (g\circ f)(x) = \frac{1}{\pi}\tan^{-1}x + 1$ and $h:\mathbb{R}\to(0,1)$ as a bijection. $\therefore$ $(0,1)$ uncountable. □