MA 3053 Practice Exam 1 solutions

1.) let $f: X \to Y$, $g: Y \to Z$. If $g \circ f$ is 1-1. Prove f is 1-1.

pf:
let $x, y \in X$ and $f(x) = f(y) \Rightarrow g(f(x)) = g(f(y)) \Rightarrow (g \circ f)(x) = (g \circ f)(y)$. But $g \circ f$ is 1-1
$\therefore$ $x = y$. Hence f is 1-1. $\square$

2.) let $f: X \to Y$. Given fncs $g, h: W \to X$ s.t. whenever $f \circ g = f \circ h$, then $g = h$. Prove f is 1-1.

pf:
consider for $w_1, w_2 \in W$, $g(w_1)$, $h(w_2)$. since
$g: W \to X$ and $h: W \to X \Rightarrow g(w_1), g(w_2) \in X$. $\therefore$ can assign $x = g(w_1)$ and $y = h(w_2)$
Now consider $f(x) = f(y) \Rightarrow f(g(w_1)) = f(h(w_2)) \Rightarrow (f \circ g)(w_1) = (f \circ h)(w_2)$. But by hyp.
$\Rightarrow g(w_1) = h(w_2) \Rightarrow x = y$. $\therefore$ f is 1-1 $\square$.

3.) Let $f: X \to Y$ and $P_\alpha \subseteq Y$ for all $\alpha \in A$. Prove $f^{-1}\left(\bigcup_{\alpha \in A} P_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(P_\alpha)$.

pf:
(a) let $x \in f^{-1}\left(\bigcup_{\alpha \in A} P_\alpha\right) \Rightarrow f(x) \in \bigcup_{\alpha \in A} P_\alpha \Rightarrow f(x) \in P_\alpha$ for some $\alpha \in A$. $\Rightarrow x \in f^{-1}(P_\alpha)$ for some $\alpha \in A$.
$\Rightarrow x \in \bigcup_{\alpha \in A} f^{-1}(P_\alpha)$. $\therefore$ $f^{-1}\left(\bigcup_{\alpha \in A} P_\alpha\right) \subseteq \bigcup_{\alpha \in A} f^{-1}(P_\alpha)$.

(b) let $x \in \bigcup_{\alpha \in A} f^{-1}(P_\alpha) \Rightarrow x \in f^{-1}(P_\alpha)$ for some $\alpha \in A \Rightarrow f(x) \in P_\alpha$ for some $\alpha \in A \Rightarrow f(x) \in \bigcup_{\alpha \in A} P_\alpha$
$\Rightarrow x \in f^{-1}\left(\bigcup_{\alpha \in A} P_\alpha\right)$. $\therefore$ $\bigcup_{\alpha \in A} f^{-1}(P_\alpha) \subseteq f^{-1}\left(\bigcup_{\alpha \in A} P_\alpha\right)$. $\therefore$ by (a), (b) set equality occurs. $\square$

4.) let $f: X \to Y$, $P_\alpha \subseteq X$ for all $\alpha \in A$. Show $f\left(\bigcup_{\alpha \in A} P_\alpha\right) = \bigcup_{\alpha \in A} f(P_\alpha)$.

pf:
(a) let $f(x) \in f\left(\bigcup_{\alpha \in A} P_\alpha\right) \Rightarrow x \in \bigcup_{\alpha \in A} P_\alpha \Rightarrow x \in P_\alpha$ for some $\alpha \in A$. $\Rightarrow f(x) \in f(P_\alpha)$ for some $\alpha \in A$
$\Rightarrow f(x) \in \bigcup_{\alpha \in A} f(P_\alpha)$. $\therefore$ $f\left(\bigcup_{\alpha \in A} P_\alpha\right) \subseteq \bigcup_{\alpha \in A} f(P_\alpha)$.

(b) let $f(x) \in \bigcup_{\alpha \in A} f(P_\alpha) \Rightarrow f(x) \in f(P_\alpha)$ for some $\alpha \in A \Rightarrow x \in P_\alpha$ for some $\alpha \in A \Rightarrow x \in \bigcup_{\alpha \in A} P_\alpha$
$\Rightarrow f(x) \in f\left(\bigcup_{\alpha \in A} P_\alpha\right)$. $\therefore$ $\bigcup_{\alpha \in A} f(P_\alpha) \subseteq f\left(\bigcup_{\alpha \in A} P_\alpha\right)$. $\therefore$ by (a), (b) the sets are equal $\square$

5.) let f be real ~~valued~~ fnc s.t. if $x_1 < x_2$ then $f(x_1) > f(x_2)$. Prove for all $b \in \mathbb{R}$. $f^{-1}(b)$ is empty or has one element.

pf:
s'pose there are $x \neq y$ s.t. $x, y \in f^{-1}(b) \Rightarrow f(x) = b$ and $f(y) = b$. $\Rightarrow f(x) = f(y)$
since $x \neq y \Rightarrow x < y$ or $x > y$ and since f is strictly have $f(x) > f(y)$ or $f(x) < f(y)$
which is impossible unless $f^{-1}(b) = \emptyset$ or $x = y \Rightarrow f^{-1}(b) = \{x\}$ $\square$

6.) let f be a real func. that is decreasing. Is it possible for f to not be 1-1?

pf:

Yes. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 1-x & x \geq 1 \\ 0 & -1 \leq x \leq 1 \\ -(x+1) & x \leq -1 \end{cases}$, then f is piecewise linear

has the property that if $x_1 \leq x_2$ then $f(x_1) \geq f(x_2)$, i.e. f is decreasing

since for $x \geq 1$ clearly $\searrow$ as for $x \leq -1$. and it's const. in $(-1, 1]$, but $f(1) = f(-1) = 0$

so not 1-1. $\square$

7.) let $\sim$ be a relation on $\mathbb{Z} \times \mathbb{Z}$ by $(a,b) \sim (c,d)$ iff $a+d = b+c$. which of 4 does it have?

pf:

It's reflexive since $(a,b) \sim (a,b)$ means $a+b = b+a$ and addition is commutative

It's symmetric since if $(a,b) \sim (c,d) \Rightarrow a+d = b+c \Rightarrow c+b = d+a \Rightarrow (c,d) \sim (a,b)$.

It's also transitive. since if $(a,b) \sim (c,d)$ and $(c,d) \sim (x,y) \Rightarrow \begin{cases} a+d = b+c \\ c+y = d+x \end{cases} \Rightarrow c = d+x-y$

$\Rightarrow a+d = d+x-y+b \Rightarrow a+y = b+x \Rightarrow (a,b) \sim (x,y)$.

Not antisym: since $(1,1) \sim (0,0)$ and $(0,0) \sim (1,1)$ but $(0,0) \neq (1,1)$ $\square$

8) let $\mathcal{F}$ be a family of sets. let R be a relation on $\mathcal{F}$ by XRY iff $X \subsetneq Y$. ~~what p~~ which 4 properties does R possess.

pf:

It's transitive since if XRY and $YRZ \Rightarrow X \subsetneq Y$ and $Y \subsetneq Z$ so $X \subsetneq Z$.

not refl. since XRX would mean $X \subsetneq X$ impossible.

not sym. since $XRY \Rightarrow YRX$ mean $X \subsetneq Y \Rightarrow Y \subsetneq X$. take $X = \mathbb{N}$ and $Y = \mathbb{R}$

clear $\mathbb{N} \subsetneq \mathbb{R}$ but $\mathbb{R} \subsetneq \mathbb{N}$ is clearly false.

not antisym. since $X \subsetneq Y$ and $Y \subsetneq X$ cannot happen simultaneously. $\square$.