MA 3053 Practice Exam 2 Solutions

1.) Let $f: X \to Y$ define relation $\sim$ on X by $x \sim y$ iff $f(x)=f(y)$. Prove $\sim$ is equiv. relation.

pf: refl: $x\sim x$ means $f(x)=f(x)$, always true since f is a well-def fnc. For sym: consider $x\sim y$ $\Rightarrow f(x)=f(y) \Rightarrow f(y)=f(x)$ so $y\sim x$. For trans: let $x\sim y$ and $y \sim z \Rightarrow f(x)=f(y)$ and $f(y)=f(z)$ so $f(x)=f(y)=f(z) \Rightarrow f(x)=f(z)$ so $x\sim z$. □

2.) Let $\mathcal{F}$ be a family of sets. Define relation $\preceq$ on $\mathcal{F}$ by $X\preceq Y$ iff $X\subseteq Y$. Show $\preceq$ is partial order.

pf: refl: $X\preceq X$ means $X\subseteq X$, always true as a set is always a subset of itself. For antisym: let $X\preceq Y$ and $Y\preceq X \Rightarrow X\subseteq Y$ and $Y\subseteq X \Rightarrow X=Y$. For trans: let $X\preceq Y$ and $Y\preceq Z$, then $X\subseteq Y$ and $Y\subseteq Z \Rightarrow X\subseteq Y\subseteq Z \Rightarrow X\subseteq Z$. $\therefore X\preceq Z$. □

3.) Prove $\sqrt{7}$ is irrational.

pf: S'pose $\sqrt{7}$ is rat'l. then $\exists p,q\in\mathbb{Z}$, $q\neq 0$, and no common factors among them s.t. $\sqrt{7} = \frac{p}{q}$
$\Rightarrow 7=\frac{p^2}{q^2} \Rightarrow p^2=7q^2 \Rightarrow p^2 \bmod 7 = 0 \Rightarrow (p \bmod 7)^2=0 \Rightarrow p\bmod 7=0 \Rightarrow 7\mid p$
so $\exists k\in\mathbb{Z}$ s.t. $p=7k \Rightarrow (7k)^2=7q^2 \Rightarrow q^2=7k^2 \Rightarrow q^2 \bmod 7=0 \Rightarrow (q\bmod 7)^2=0 \Rightarrow q \bmod 7=0$
so $7\mid q \Rightarrow p,q$ have common factors contradiction. $\therefore \sqrt{7}$ is irrational. □

4.) Prove $\exists$ irrat'l #s x,y s.t. $x^y\in\mathbb{Q}$.

pf: consider $\sqrt{2}^{\sqrt{2}}$ and $\sqrt{2}$. Know $\sqrt{2}$ is irrat'l. Then either $\sqrt{2}^{\sqrt{2}}$ is irrat'l or rat'l. If $\sqrt{2}^{\sqrt{2}}$ rat'l done. Otherwise $\sqrt{2}^{\sqrt{2}}$ is irrat'l then $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2\in\mathbb{Q}$. □

5.) Prove $\forall n\in\mathbb{N}$, $(x+y)^n=\sum_{k=0}^n \binom{n}{k}x^{n-k}y^k$.

pf: 1st base case $n=0$: then LHS $=(x+y)^0=1$, RHS $=\sum_{k=0}^0\binom{0}{k}x^{-k}y^k=1$. so LHS=RHS.

Inductive step: S'pose statement is true for n and consider $(x+y)^{n+1}=(x+y)(x+y)^n=(x+y)\sum_{k=0}^n\binom{n}{k}x^{n-k}y^k =$
$=\sum_{k=0}^n \binom{n}{k}x^{n+1-k}y^k+\sum_{k=0}^n\binom{n}{k} x^{n-k}y^{k+1} := I + II$. via assumption. Then in II let $j=k+1$
$\Rightarrow II = \sum_{j=1}^{n+1}\binom{n}{j-1}x^{n-(j-1)}y^j = \sum_{j=1}^{n+1}\binom{n}{j-1}x^{n+1-j}y^j$. In I set $k=j$ so now $(x+y)^{n+1}=I+II=$
$=\sum_{j=0}^n\binom{n}{j} x^{n+1-j}y^j+\sum_{j=1}^{n+1}\binom{n}{j-1}x^{n+1-j}y^j = \binom{n}{0} x^{n+1}+\sum_{j=1}^n\binom{n}{j} x^{n+1-j}y^j+\sum_{j=1}^n\binom{n}{j-1}x^{n+1-j}y^j+\binom{n}{n} y^{n+1}$
$=\binom{n}{0} x^{n+1}+\sum_{j=1}^n\left[\binom{n}{j}+\binom{n}{j-1}\right]x^{n+1-j}y^j+\binom{n}{n} y^{n+1} = \binom{n+1}{0}x^{n+1}+\sum_{j=1}^n\binom{n+1}{j}x^{n+1-j}y^j+\binom{n+1}{n+1}y^{n+1}$
$=\sum_{j=0}^{n+1}\binom{n+1}{j}x^{n+1-j}y^j$ which the $(n+1)$st case. $\therefore$ by induction hyp. true $\forall n\in\mathbb{N}$. □

6.) Prove $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall n \in \mathbb{N}$

pf:
1^{st} base case $n=0$: $LHS = \sum_{k=0}^{0} k^2 = 0$, $RHS = \frac{0(0+1)(0+1)}{6} = 0$ so $RHS = LHS$.

Inductive step: s'pose statement true for n. consider $\sum_{k=0}^{n+1} k^2 = (n+1)^2 + \sum_{k=1}^{n} k^2 = (n+1)^2 + \frac{n(n+1)(2n+1)}{6}$

$= \frac{(n+1)}{6}\left(6(n+1) + n(2n+1)\right) = \frac{n+1}{6}\left(6(n+1) + 2n^2+n\right) = \frac{n+1}{6}(n+2)(2n+3) = \underline{\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}}$

which is $(n+1)^{st}$ case. $\therefore$ by inductive hyp true $\forall n \in \mathbb{N}$. □

7.) Prove $2^n > n$ $\forall n \in \mathbb{N}$.

pf:
1^{st} base case $n=0$: $LHS = 2^0 = 1$ & $RHS = 0$ clear $LHS > RHS$.

Inductive step: s'pose statement true for n. consider $2^{n+1} = 2 \cdot 2^n > 2n > n+1$ if $n \geq 1$

$\Rightarrow 2^{n+1} > n+1$ which is $(n+1)^{st}$ case. $\therefore$ by inductive hyp. true $\forall n \in \mathbb{N}$. □

8.) Let P, Q be statements, Prove $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

pf:

P	Q	$\neg P$	$\neg P \vee Q$	$P \Rightarrow Q$
0	0	1	1	1
1	0	0	0	0
0	1	1	1	1
1	1	0	1	1

so clear last two columns are equivalent. □