MA 3053 Practice ~~Final~~ Final Exam Solutions

1.) Prove that $(f+g)'(a) = f'(a)+g'(a)$, provided $f'(a), g'(a)$ exist.

pf:
S'pose $f'(a), g'(a)$ exist. Then $(f+g)'(a) = \lim_{x\to a} \frac{(f+g)(x) - (f+g)(a)}{x-a} = \lim_{x\to a} \frac{f(x)+g(x)-f(a)-g(a)}{x-a} =$
$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} + \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(a)+g'(a)$ by limit laws □

2.) Let $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$. Prove $\lim_{x\to a} f(x)g(x) = LM$

pf:
Since limits of f and g exist at $x=a$, know $\forall \varepsilon>0$, $\exists \delta_1>0$ s.t. if $0<|x-a|<\delta_1$, then $|f(x)-L| < \frac{\varepsilon}{2(|M|+1)}$
and $\exists \delta_2>0$ s.t. if $0<|x-a|<\delta_2$, then $|g(x)-M| < \frac{\varepsilon}{2(|L|+1)}$. Pick $\delta = \min\{\delta_1, \delta_2\}$. Then if $|x-a|<\delta$
$\Rightarrow |f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| \le |f(x)||g(x)-M| + |M||f(x)-L| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ □.

3.) Prove $aX = \{ax : x\in X\}$ is countable if X is.

pf:
Since X is countable $\exists f: \mathbb{N}\to X$, bijection. Consider $g: X\to aX$ via $g(x)=ax$. Then given any $y\in aX$
y has form $y=az$ w/ $z\in X$. So $g(x)=y \Rightarrow ax=az \Rightarrow x=z\in X$ since $a\neq 0$.
So g is onto. S'pose $g(x)=g(y) \Rightarrow ax=ay \Rightarrow x=y$ so g is 1-1. So g is a bijection.
Then define $h: \mathbb{N}\to aX$ via $h = g\circ f$ so h bijection as comp. of bij. is bije. $\therefore aX$ countable □

4.) Prove $X+a = \{x+a : x\in X\}$ is countable if X is.

pf:
Since X countable, $\exists f: \mathbb{N}\to X$, bijection. Consider $g: X\to aX$ via $g(x)=x+a$. Then given any $y\in X+a$
y has form $y=z+a$ w/ $z\in X$. So $g(x)=y \Rightarrow a+x=a+z \Rightarrow x=z\in X$. So g is onto.
Now s'pose $g(x)=g(y) \Rightarrow a+x=a+y \Rightarrow x=y$ so g is 1-1. So g is bijection. Define $h: \mathbb{N}\to X+a$
via $h=g\circ f$, so h biject. Like in (3) so $X+a$ is countable □

5) Let $Y\subseteq X$ and X countable. Prove Y countable.

pf:
Since X countable, $\exists f: \mathbb{N}\to X$ bijection. Define $g: \mathbb{N}\to Y$ via $g = f|_Y$. Then if $g(x)=g(y)$
$\Rightarrow f|_Y(x) = f|_Y(y)$ but f is 1-1 $\Rightarrow x=y$. So g is 1-1. Now given any $y\in Y$ consider $g(x)=y$
$\Rightarrow f|_Y(x)=y$ but f is onto $\Rightarrow x\in\mathbb{N}$. $\therefore$ g is onto. So g is biject. $\Rightarrow$ Y countable. □

6.) Let $Y\subseteq X$ and Y uncountable. Prove X is uncountable.

pf:
S'pose X is countable. $\therefore$ by (5) $\Rightarrow$ Y is countable. contradiction, as Y uncountable.
$\therefore$ X must be uncountable. □

7.) let $d = \gcd(a,b)$ and $a = da'$, $b = db'$. Prove $\gcd(a', b') = 1$.

pf:

Since ~~gcd~~ $d = \gcd(a,b)$, $\exists\, m,n \in \mathbb{Z}$ s.t. $d = ma + nb \Rightarrow d = mda' + ndb' \Rightarrow 1 = ma' + nb'$

$\therefore \gcd(a', b') = 1$. $\square$

8.) let $a,b,c \in \mathbb{Z}$. If $\gcd(a, bc) = 1$, prove $\gcd(a,b) = 1$ and $\gcd(a,c) = 1$.

pf:

since $\gcd(a, bc) = 1$, $\exists\, m,n \in \mathbb{Z}$ s.t. $1 = ma + n(bc)$. $\Rightarrow 1 = ma + (nb)c$ but $nb \in \mathbb{Z}$

$\Rightarrow \gcd(a,c) = 1$. OTH. $1 = ma + n(bc) \Rightarrow 1 = ma + (nc)b$, but $nc \in \mathbb{Z} \Rightarrow \gcd(a,b) = 1$ $\square$