Practice Exam 2 Solutions

1.) $C^1(\mathbb{R}) = \{ f(x) : f'(x) \text{ exists} \}$

let $f, g \in C^1(\mathbb{R})$ consider $\frac{d}{dx}(f+g) = \underbrace{\frac{df}{dx} + \frac{dg}{dx}}_{\text{both exist}}$ by calculus so $\frac{d}{dx}(f+g)$ exists

$\Rightarrow f+g \in C^1(\mathbb{R})$.

let a scalar then consider $\frac{d}{dx}(af) = a \underbrace{\frac{df}{dx}}_{\text{exists}}$ so $\frac{d}{dx}(af)$ exists $\Rightarrow af \in C^1(\mathbb{R})$

$\frac{d}{dx}(0) = 0$ always exists so $C^1(\mathbb{R})$ is a subspace.

2.) $B(\mathbb{R}) = \{ f(x) : f(a) = 0 \text{ for fixed } a \in \mathbb{R} \}$

let $f, g \in B(\mathbb{R})$ consider $(f+g)(a) = f(a) + g(a) = 0 + 0 = 0$ so $f+g \in B(\mathbb{R})$

let c be scalar then $(cf)(a) = c f(a) = c \cdot 0 = 0$ so $cf \in B(\mathbb{R})$

$0(a) = 0$ always so $0 \in B(\mathbb{R})$ so $B(\mathbb{R})$ is a subspace

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3.) $S = \{ t^2+1,\ t^2+t,\ t+1 \}$

notice $\dim S = 3$ and $\dim P_2 = 3$, just need to check if S is lin indep.

consider $c_1(t^2+1) + c_2(t^2+t) + c_3(t+1) = 0 \Rightarrow (c_1+c_2)t^2 + (c_2+c_3)t + (c_1+c_3)1 = 0$

$$\Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \\ c_1 + c_3 = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow c_1 = c_2 = c_3 = 0$$

so S lin indp. hence S is basis for P_2

4.) $S = \{ 2t^2+1,\ 3t^2 - 4t,\ 4t+9 \}$

like #3 $\dim S = 3 = \dim P_2 = 3$, just check if S is lin. indep.

so $c_1(2t^2+1) + c_2(3t^2-4t) + c_3(4t+9) = 0 \Rightarrow (2c_1+3c_2)t^2 + (-4c_2+4c_3)t + (c_1+9c_3)\cdot 1 = 0$

$$\Rightarrow \begin{cases} 2c_1 + 3c_2 = 0 \\ -4c_2 + 4c_3 = 0 \\ c_1 + 9c_3 = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 1 & 0 & 9 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 9 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ get free variable

so S is lin dep. $\Rightarrow S$ is not a basis for P_2

5.) $T: P_2 \to \mathbb{R}$

$$T(p) = \int_0^1 (a_0 + a_1 t + a_2 t^2)\, dt = \left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3}\right)\Big|_0^1 = a_0 + \frac{a_1}{2} + \frac{a_2}{3}$$

then $\ker(T) = \{p : T(p) = 0\}$ so $T(p) = 0 \Rightarrow a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0 \Rightarrow a_0 = -\frac{1}{2}a_1 - \frac{1}{3}a_2$

so $p(t) = -\frac{1}{2}a_1 - \frac{1}{3}a_2 + a_1 t + a_2 t^2 = a_1\left(t - \frac{1}{2}\right) + a_2\left(t^2 - \frac{1}{3}\right)$

so $\ker(T) = \text{span}\left\{ t - \frac{1}{2},\ t^2 - \frac{1}{3} \right\}$

6.) $T: P_2 \to P_1$

$T(p) = \frac{d}{dt}(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$ want all p s.t. $T(p) = 0$ for all t

$\Rightarrow a_1 = 2a_2 = 0$ so $p(t) = a_0$ so $\ker(T) = \text{span}\{1\}$

7.) A is 10×10, then $d = \dim \ker(A)$, so $d^2 - 9d = (d-9)d = 0$

$\Rightarrow d = 0$ or $d = 9$ if $d = 9 \Rightarrow \dim \ker(A) = 9$ so by rk null thm

$rk(A) = 10 - 9 = 1$ if $d = 0 \Rightarrow \dim \ker(A) = 0$ again by ↑

$rk(A) = 10 - 0 = 10$ so ~~dirc~~ dim col $= 1$ or 10

if $rk(A) = 10 \Rightarrow A$ is invertible

8.) A $m \times n$, then A^t is $n \times m$ so $A^t A$ is $n \times n$ if $A^t A$ is nonsingular

$\Rightarrow A^t A x = 0 \Rightarrow x = 0$ always so $A^t(Ax) = 0 \Rightarrow Ax = 0$ and $x = 0$ always

i.e. $Ax = 0$ has only trivial soln so $\dim \ker(A) = 0$ by rk-null thm

$\Rightarrow rk(A) = n - 0 = n$

9.) consider $Ax = x \Rightarrow$ ~~(A-I_n)x=0~~ $Ax - x = 0 \Rightarrow (A - I_n)x = 0$

but no nonzero vector solves $Ax = x \Rightarrow$ no nonzero vector solves ↑ i.e. $x = 0$

is only soln so $\dim \ker(A - I_n) = 0$ by rk-null thm $\Rightarrow rk(A - I_n) = n$

$\Rightarrow A - I_n$ is invertible.