Practice Exam 3 Solutions

1.) $T: P_2 \to P_2$, $T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$

so $T(1) = 3 + 5t$, $T(t) = -2t + 4t^2$, $T(t^2) = t^2$

$[T(1)]_B = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$, $[T(t)]_B = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$, $[T(t^2)]_B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

so $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{pmatrix}$, then $P_A(\lambda) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 5 & -2-\lambda & 0 \\ 0 & 4 & 1-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda)(1-\lambda)$

so $P_A(\lambda) = 0 \Rightarrow \lambda = -2, 1, 3$

$\lambda = -2$: $(A + 2I)v = 0 \Rightarrow \left(\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} v_3$ pick $v_2 = 1$ so $v = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}$

$\lambda = 1$: $(A - 1I)v = 0 \Rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 5 & -3 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_3$ pick $v_3 = 1$ so $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 3$: $(A - 3I)v = 0 \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 5 & -5 & 0 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right) \xrightarrow{\text{ref}} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} v_3$ pick $v_3 = 1 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

so $\lambda = -2$, $p(t) = 3t - 4t^2$, $\lambda = 1$, $p(t) = t^2$, $\lambda = 3$, $p(t) = 1 + t + 2t^2$

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2) $T: P_1 \to P_1$, $T(a_0 + a_1 t) = (2a_0 + 7a_1) + (7a_0 + 2a_1)t$ so $T(1) = 2 + 7t$, $T(t) = 7 + 2t$

so $[T(1)]_B = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$, $[T(t)]_B = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$ and $P_A(\lambda) = \begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 49$

so $P_A(\lambda) = 0 \Rightarrow (2-\lambda)^2 - 49 = 0 \Rightarrow (2-\lambda)^2 = 49 \Rightarrow 2 - \lambda = \pm 7 \Rightarrow \lambda = 2 \pm 7 = 9, -5$

$\lambda = 9$: $(A - 9I)v = 0 \Rightarrow \left(\begin{array}{cc|c} -7 & 7 & 0 \\ 7 & -7 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_2$ pick $v_2 = 1$ so $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -5$: $(A + 5I)v = 0 \Rightarrow \left(\begin{array}{cc|c} 7 & 7 & 0 \\ 7 & 7 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} v_2$ pick $v_2 = 1$ so $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so $\lambda = 9$: $p(t) = 1 + t$, $\lambda = -5$: $p(t) = 1 - t$

3.) $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ so $p_A(\lambda) = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = 3 - 3\lambda - \lambda + \lambda^2 + 2$
$= \lambda^2 - 4\lambda + 5$

$p_A(\lambda) = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm 2i$

$\lambda = 2-2i$: $(A - (2-2i)I)v = 0 \Rightarrow \left(\begin{array}{cc|c} -1+2i & -2 & 0 \\ 1 & 1+2i & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{cc|c} 1 & 1+2i & 0 \\ 0 & 0 & 0 \end{array}\right)$ so $v = \begin{pmatrix} -1 \\ 1+2i \end{pmatrix} v_2$

pick $v_2 = 1$

$v = \begin{pmatrix} -1 \\ 1+2i \end{pmatrix}$

for $\lambda = 2+2i$: $v = \overline{\begin{pmatrix} -1 \\ 1+2i \end{pmatrix}} = \begin{pmatrix} -1 \\ 1-2i \end{pmatrix}$

4.) $A = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix}$ so $p_A(\lambda) = \begin{vmatrix} 3-\lambda & 1 \\ -2 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda) + 2 = 15 - 8\lambda + \lambda^2 + 2 = \lambda^2 - 8\lambda + 17$

$p_A(\lambda) = 0 \Rightarrow \lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm 2i$

$\lambda = 4-2i$: $(A - (4-2i)I)v = 0 \Rightarrow \left(\begin{array}{cc|c} -1+2i & 1 & 0 \\ -2 & 1+2i & 0 \end{array}\right) \xrightarrow{ref} \left(\begin{array}{cc|c} -2 & 1+2i & 0 \\ 0 & 0 & 0 \end{array}\right)$ so $v = \begin{pmatrix} 2 \\ 1+2i \end{pmatrix} v_2$

pick $v_2 = 1$ so $v = \begin{pmatrix} 2 \\ 1+2i \end{pmatrix}$

for $\lambda = 4+2i$: $v = \overline{\begin{pmatrix} 2 \\ 1+2i \end{pmatrix}} = \begin{pmatrix} 2 \\ 1-2i \end{pmatrix}$

5.) $T(x) = \frac{\langle x, u \rangle}{\|u\|^2} u = \frac{x \cdot u}{\|u\|^2} u$

so (1) $T(x+y) = \frac{(x+y) \cdot u}{\|u\|^2} u = \frac{1}{\|u\|^2} (x \cdot u + y \cdot u) u = \frac{x \cdot u}{\|u\|^2} u + \frac{y \cdot u}{\|u\|^2} u = T(x) + T(y)$

(2) for c - scalar $T(cx) = \frac{(cx) \cdot u}{\|u\|^2} u = \frac{c(x \cdot u)}{\|u\|^2} u = cT(x)$

so T is linear transform

6) $T(x) = \sum_{j=1}^{k} \frac{\langle x, v_j \rangle}{\|v_j\|^2} v_j$

(1) pick $x, y \in \mathbb{R}^n$ then $T(x+y) = \sum_{j=1}^{k} \frac{\langle x+y, v_j \rangle}{\|v_j\|^2} v_j = \sum_{j=1}^{k} \left(\frac{\langle x, v_j \rangle + \langle y, v_j \rangle}{\|v_j\|^2} \right) v_j = \sum_{j=1}^{k} \frac{\langle x, v_j \rangle}{\|v_j\|^2} v_j + \sum_{j=1}^{k} \frac{\langle y, v_j \rangle}{\|v_j\|} v_j$

$= T(x) + T(y)$

(2) for scalar c, $T(cx) = \sum_{j=1}^{k} \frac{\langle cx, v_j \rangle}{\|v_j\|^2} v_j = \sum_{j=1}^{k} \frac{c \langle x, v_j \rangle}{\|v_j\|^2} v_j = c \sum_{j=1}^{k} \frac{\langle x, v_j \rangle}{\|v_j\|^2} v_j = c\,T(x)$

so T linear.

now we want x : $T(x) = 0 \Rightarrow \text{Proj}_W(x) = 0 \Rightarrow \langle x, v_j \rangle = 0$ for all $j = 1, \dots, k$

$\Rightarrow x \in W^\perp$ so $\ker(T) = W^\perp$ Think of the matrix formed by the eq.

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7) $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

set $w_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ then $w_2 = v_2 - \text{Proj}_{w_1} v_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\|w_1\|^2} w_1$, $\langle w_1, v_2 \rangle = 1+1+1 = 3$, $\|w_1\|^2 = 1+1+1+1 = 4$

so $w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ so $w_3 = v_3 - \text{Proj}_{w_1} v_3 - \text{Proj}_{w_2} v_3$

$\langle w_1, v_3 \rangle = 1+1 = 2$, $\langle w_2, v_3 \rangle = \frac{1}{4}(1+1) = \frac{1}{2}$, $\|w_2\|^2 = \frac{1}{16}(9+1+1+1) = \frac{12}{16} = \frac{3}{4}$

so $w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{4} \cdot \frac{4}{3} \cdot \frac{1}{2} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ $\|w_3\|^2 = \frac{4}{9}$

then $u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \frac{w_2}{\|w_2\|} = \frac{2}{\sqrt{3}} \cdot \frac{1}{4} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{3}} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $u_3 = \frac{w_3}{\|w_3\|} = \frac{3}{2} \cdot \frac{2}{3} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

8.) $v_1 = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3 \\ 14 \\ -7 \end{pmatrix}$

set $w_1 = v_1 = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ then $w_2 = v_2 - \operatorname{Proj}_{w_1} v_2 = v_2 - \dfrac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$

$\langle v_2, w_1 \rangle = -9 - 48 - 35 = 92$, $\|w_1\|^2 = 9 + 16 + 25 = 50$

so $w_2 = \begin{pmatrix} -3 \\ 14 \\ -7 \end{pmatrix} - \dfrac{92}{50}\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} = \dfrac{1}{50}\begin{pmatrix} -426 \\ 332 \\ 110 \end{pmatrix} \qquad \|w_2\|^2 = \dfrac{303{,}800}{2500} = \dfrac{3{,}038}{25}$

so $u_1 = \dfrac{w_1}{\|w_1\|} = \dfrac{1}{\sqrt{50}}\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \quad u_2 = \dfrac{w_2}{\|w_2\|} = \sqrt{\dfrac{25}{3000}}\,\dfrac{1}{50}\begin{pmatrix} -426 \\ 332 \\ 110 \end{pmatrix} = \dfrac{1}{40\sqrt{193}}\begin{pmatrix} -426 \\ 332 \\ 110 \end{pmatrix}$

9.) Know P has orthonormal columns so $P^tP = I$

S'pose $\langle x, y \rangle = 0$ ie $x \cdot y = 0$ so $(Px)\cdot(Py) = (Px)^t Py = x^t P^t P y$

$= x^t y = x \cdot y = 0 \Rightarrow \langle Px, Py \rangle = 0$

now s'pose $\langle Px, Py \rangle = 0$ so $x \cdot y = x^t y = x^t I y = x^t P^t P y = (Px)^t Py$

$= \langle Px, Py \rangle = 0 \Rightarrow \langle x, y \rangle = 0$