Practice Final Solutions

1.) consider $(A^t)^2 = A^t A^t = (AA)^t = (A^2)^t = A^t$ since A is idempotent

now consider $(A+B)^2 = A^2 + AB + BA + B^2 = A + B + AB + BA \neq A+B$ in general

so $A+B$ is not idempotent in general

2.) know $A^k = 0$ for some $k \geq 1$ so $I_n = I_n - A^k = (I_n - A)(A^{k-1} + A^{k-2} + \cdots + A + I_n)$

$\Rightarrow$ $I_n - A$ is invertible on left and $I_n - A^k = (A^{k-1} + \cdots + A + I_n)(I_n - A)$

so $I_n - A$ invertible on right so $I_n - A$ invertible

and $(I_n - A)^{-1} = \sum_{j=1}^{k-1} A^j = A^{k-1} + A^{k-2} + \cdots + A + I_n$

3.) have $S = \{-t^2 + t + 2,\ 2t^2 + 2t + 3,\ 4t^2 - 1\}$ notice S has 3 vectors spanning

and $\dim P_2 = 3$, so S could be a basis, just check if S lin indep.

consider $c_1(-t^2 + t + 2) + c_2(2t^2 + 2t + 3) + c_3(4t^2 - 1) = 0$

$\Rightarrow (-c_1 + 2c_2 + 4c_3)t^2 + (c_1 + 2c_2)t + (2c_1 + 3c_2 - c_3) \cdot 1 = 0$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 2 & 4 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

a row of zeros $\Rightarrow$ S is dep.

so S is not a basis

4.) have $S = \{t^2 + 1,\ 3t^2 + 2t + 1,\ 6t^2 + 6t + 3\}$ S has 3 spanning vectors

and $\dim P_2 = 3$, so just check if S is lin indep. or not

consider $c_1(t^2 + 1) + c_2(3t^2 + 2t + 1) + c_3(6t^2 + 6t + 3) = 0$

$\Rightarrow (c_1 + 3c_2 + 6c_3)t^2 + (2c_2 + 6c_3)t + (c_1 + c_2 + 3c_3) \cdot 1 = 0$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 6 & 0 \\ 0 & 2 & 6 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\Rightarrow c_1 = c_2 = c_3 = 0$ so S is lin indep

so S is a basis

5.) $T(p) = \frac{dp}{dt}$, $T(1) = 0$, $T(t) = 1$, $T(t^2) = 2t$, $T(t^3) = 3t^2$, $T(t^4) = 4t^3$, $T(t^5) = 5t^4$

so $[T(1)]_B = \begin{pmatrix} 0\\0\\0\\0\\0\\0 \end{pmatrix}$, $[T(t)]_B = \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}$, $[T(t^2)]_B = \begin{pmatrix} 0\\2\\0\\0\\0\\0 \end{pmatrix}$, $[T(t^3)]_B = \begin{pmatrix} 0\\0\\3\\0\\0\\0 \end{pmatrix}$, $[T(t^4)]_B = \begin{pmatrix} 0\\0\\0\\4\\0\\0 \end{pmatrix}$, $[T(t^5)]_B = \begin{pmatrix} 0\\0\\0\\0\\5\\0 \end{pmatrix}$

so $A = \begin{pmatrix} 0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&3&0&0\\0&0&0&0&4&0\\0&0&0&0&0&5\\0&0&0&0&0&0 \end{pmatrix}$ rep. T and $\ker(T) = \ker(A)$

but $\mathrm{rref}(A) = \begin{pmatrix} 0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0 \end{pmatrix}$ so $Ax = 0 \Rightarrow x = \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}$ so $p(t) = c \Leftarrow$ const

so $\ker(T) = \mathrm{span}\{1\}$

6.) $T(p) = \int_0^t p(\tau)\,d\tau$ $T(1) = \int_0^t d\tau = t$, $T(t) = \int_0^t \tau\,d\tau = \frac{t^2}{2}$, $T(t^2) = \frac{t^3}{3}$, $T(t^3) = \frac{t^4}{4}$

so $[T(1)]_{B_2} = \begin{pmatrix} 0\\1\\0\\0\\0 \end{pmatrix}$, $[T(t)]_{B_2} = \begin{pmatrix} 0\\0\\\frac{1}{2}\\0\\0 \end{pmatrix}$, $[T(t^2)]_{B_2} = \begin{pmatrix} 0\\0\\0\\\frac{1}{3}\\0 \end{pmatrix}$, $[T(t^3)]_{B_2} = \begin{pmatrix} 0\\0\\0\\0\\\frac{1}{4} \end{pmatrix}$

So $A = \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{3}&0\\0&0&0&\frac{1}{4} \end{pmatrix}$ rep T, and $\ker(T) = \ker(A)$ but $\mathrm{rref}(A) = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}$

and A is 5×4 so $Ax = 0 \Rightarrow x = 0$ only

so $p(t) = 0$ and $\ker(T) = \{0\}$

7.) $A = \begin{pmatrix} 0&0&3\\1&0&-1\\0&1&3 \end{pmatrix}$ so $p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda&0&3\\1&-\lambda&-1\\0&1&3-\lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 + \lambda - 3 = (\lambda-3)(\lambda^2+1)$

so the eigenvalues are $\lambda = 3, i, -i$

$\underline{\lambda = 3:}$ $(A - 3I)v = 0 \Rightarrow \left(\begin{array}{ccc|c} -3&0&3&0\\1&-3&-1&0\\0&1&0&0 \end{array}\right) \xrightarrow{\mathrm{rref}} \left(\begin{array}{ccc|c} 1&0&-1&0\\0&1&0&0\\0&0&0&0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1\\0\\1 \end{pmatrix}$

$\underline{\lambda = i:}$ $(A - iI)v = 0 \Rightarrow \left(\begin{array}{ccc|c} -i&0&3&0\\1&-i&-1&0\\0&1&3-i&0 \end{array}\right) \xrightarrow{\mathrm{rref}} \left(\begin{array}{ccc|c} 1&0&3i&0\\0&1&3-i&0\\0&0&0&0 \end{array}\right) \Rightarrow v = \begin{pmatrix} -3i\\i-3\\1 \end{pmatrix}$

$\underline{\lambda = -i:}$ know $v = \overline{\begin{pmatrix} -3i\\i-3\\1 \end{pmatrix}} = \begin{pmatrix} 3i\\-i-3\\1 \end{pmatrix}$

8.) $A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix}$ so $P_A(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 2 & 2-\lambda & -2 \\ 3 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(\lambda+1)(\lambda-4)$

so the eigenvalues are $\lambda = -1, 2, 4$

$\underline{\lambda=-1}$: $(A+I)v=0 \Rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 2 & 3 & -2 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & \frac{24}{21} & 0 \\ 0 & 1 & -\frac{10}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} -24 \\ 30 \\ 1 \end{pmatrix}$

$\underline{\lambda=2}$: $(A-2I)v=0 \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 1 & -1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\underline{\lambda=4}$: $(A-4I)v=0 \Rightarrow \left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

9.) $S = \{\overset{v_1}{1}, \overset{v_2}{t}, \overset{v_3}{t^2}\}$ set $w_1 = 1$ so $w_2 = v_2 - Proj_{w_1}(v_2) = t - \frac{\langle t,1\rangle}{\|1\|^2}\cdot 1$

$\langle t,1\rangle = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$ $\quad \|1\|^2 = \int_0^1 1\,dt = 1$

so $w_2 = t - \frac{1}{2}$ then $w_3 = v_3 - Proj_{w_1}(v_3) - Proj_{w_2}(v_3) = t^2 - \frac{\langle t^2,1\rangle}{\|1\|^2}\cdot 1 - \frac{\langle t^2, t-\frac{1}{2}\rangle}{\|t-\frac{1}{2}\|^2}(t-\tfrac{1}{2})$

$\langle t^2,1\rangle = \int_0^1 t^2\,dt = \frac{1}{3}$, $\langle t^2, t-\frac{1}{2}\rangle = \int_0^1 (t^3 - \frac{1}{2}t^2)\,dt = \left(\frac{t^4}{4} - \frac{1}{6}t^3\right)\Big|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

and $\|t-\frac{1}{2}\|^2 = \int_0^1 (t-\frac{1}{2})^2\,dt = \int_0^1 (t^2 - t + \frac{1}{4})\,dt = \frac{1}{12}$ so $w_3 = t^2 - \frac{\frac{1}{12}}{\frac{1}{12}}(t-\frac{1}{2}) - \frac{1}{3} = t^2 - t - \frac{1}{2} - \frac{1}{3}$
$= t^2 - t - \frac{5}{6}$

then $\|t^2 - t - \frac{5}{6}\|^2 = \int_0^1 (t^2 - t - \frac{5}{6})^2\,dt = \int_0^1 (t^4 - 2t^3 - \frac{2}{3}t^2 + \frac{5}{3}t + \frac{25}{36})\,dt = \frac{9}{20}$

so $u_1 = 1$, $u_2 = \sqrt{12}(t-\frac{1}{2})$, $u_3 = \frac{\sqrt{20}}{3}(t^2 - t - \frac{5}{6})$

10.) $S = \{1, t\}$ set $w_1 = 1$ so $w_2 = t - Proj_{w_1}(t) = t - \frac{\langle t,1\rangle}{\|1\|^2}\cdot 1$

and $\langle t,1\rangle = \int_0^1 t\,dt = \frac{1}{2}$, $\|1\|^2 = 1$ so $w_2 = t - \frac{1}{2}$ and from above $\|w_2\| = \frac{1}{\sqrt{12}}$

so $u_1 = 1$, $u_2 = \sqrt{12}(t - \frac{1}{2})$

11.) $f(t) = |t|$ on $[-\pi, \pi]$ so $f(t) = \begin{cases} t & 0 < t < \pi \\ -t & -\pi < t < 0 \end{cases}$

so $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-t)dt + \int_0^{\pi} t\,dt\right) = \frac{1}{2\pi}\left(-\frac{t^2}{2}\Big|_{-\pi}^{0} + \frac{t^2}{2}\Big|_0^{\pi}\right) = \frac{\pi}{2}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} -t\cos(nt)dt + \int_0^{\pi} t\cos(nt)\,dt\right)$ via by parts $u = t$ $du = dt$

$dv = \cos(nt)dt$

$v = \frac{1}{n}\sin(nt)$

$= \frac{1}{\pi}\left(-\left[\frac{1}{n}t\sin(nt)\Big|_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin(nt)\,dt\right] + \frac{1}{n}t\sin(nt)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nt)\,dt\right)$

$= \frac{2}{\pi n^2}\left((-1)^n - 1\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{-4}{\pi n^2} & \text{if } n \text{ odd} \end{cases}$

th $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$ but ~~but that~~ $\sin(nt)$ is odd and $f(t)$ even

so $f(t)\sin(nt)$ is odd and $\int_{-a}^{a}(\text{odd func})dt = 0$ $\Rightarrow b_n = 0$

$\Rightarrow f(t) \approx -\frac{4}{\pi}\sum_{k=0}^{n}\frac{\cos[(2k+1)t]}{(2k+1)^2} + \frac{\pi}{2}$ th let $n \to \infty$ $f(t) \approx \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos[(2k+1)t]}{(2k+1)^2}$

at $t=0$ $\Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$ but $\frac{1}{k^2} = \frac{1}{(2k+1)^2} + \frac{1}{(2k)^2}$

$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} \Rightarrow \frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{8} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$

12.) $f(t) = t^2$ on $[0, 2\pi]$ so $a_0 = \frac{1}{2\pi}\int_0^{2\pi} t^2 dt = \frac{1}{2\pi}\frac{t^3}{3}\Big|_0^{2\pi} = \frac{4\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2\cos(nt)dt = \frac{1}{\pi}\left(\frac{t^2}{n}\sin(nt)\Big|_0^{2\pi} - \frac{2}{n}\int_0^{2\pi} t\sin(nt)\,dt\right)$$

$u = t^2$ $\quad dv = \cos(nt)dt$
$du = 2t\,dt$ $\quad v = \frac{1}{n}\sin(nt)$

$$= -\frac{2}{n\pi}\int_0^{2\pi} t\sin(nt)dt$$

$u = t$ $\quad du = dt$
$dv = \sin(nt)\,dt$
$v = -\frac{1}{n}\cos(nt)$

$$\text{so } a_n = \frac{2}{n^2\pi}\left(t\cos(nt)\Big|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\cos(nt)dt\right) = \frac{4}{n^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2\sin(nt)dt = \frac{1}{\pi}\left(-\frac{t^2}{n}\cos(nt)\Big|_0^{2\pi} + \frac{2}{n}\int_0^{2\pi} t\cos(nt)dt\right)$$

$u = t^2$ $\quad dv = \sin(nt)\,dt$
$du = 2t\,dt$ $\quad v = -\frac{1}{n}\cos(nt)$

$$= \frac{1}{\pi}\left(-\frac{4\pi^2}{n}(-1)^n + \frac{2}{n}\int_0^{2\pi} t\cos(nt)dt\right)$$

$u = t$ $\quad dv = \cos(nt)dt$
$du = dt$ $\quad v = \frac{1}{n}\sin(nt)$

$$\text{so } b_n = \frac{1}{\pi}\left(\frac{4\pi^2(-1)^{n+1}}{n} + \frac{2}{n}\left[\frac{t}{n}\sin(nt)\Big|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin(nt)dt\right]\right)$$

$$= \frac{1}{\pi}\left(\frac{4\pi^2(-1)^{n+1}}{n} - \frac{2}{n^3}\left((-1)^n - 1\right)\right)$$

$$\text{so } f(t) \sim \frac{4\pi^2}{3} + \sum_{k=1}^{n}\frac{4}{k^2}\left((-1)^k - 1\right)\cos(kt) + \frac{1}{\pi}\sum_{k=1}^{n}\left[\frac{4\pi^2(-1)^{k+1}}{k} - \frac{2}{n^3}\left((-1)^k - 1\right)\right]\sin(kt)$$

13.) 1^{st} notice if $u = \begin{pmatrix}\sqrt{a}\\ \sqrt{b}\end{pmatrix}$ and $v = \begin{pmatrix}\sqrt{b}\\ \sqrt{a}\end{pmatrix}$ then the CS $\leq$ says

$|\langle u, v\rangle| \leq \|u\|\,\|v\|$ but $\langle u, v\rangle = 2\sqrt{ab}$ and $\|u\| = \sqrt{a+b} = \|v\|$

$\Rightarrow 2\sqrt{ab} \leq a+b \Rightarrow \sqrt{ab} \leq \frac{1}{2}(a+b)$ can generalize this $\leq$ to n components of a vector

w/ $x_1, \ldots, x_n \geq 0$ to get $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$ $(*)$

recall $\det(A) = \lambda_1 \cdots \lambda_n$ and $tr(A) = \lambda_1 + \cdots + \lambda_n$ for eigenvalues $\lambda_1, \ldots, \lambda_n$

so $\sqrt[n]{\det(A)} = \sqrt[n]{\lambda_1\lambda_2\cdots\lambda_n} \underset{\text{via }(*)}{\leq} \frac{1}{n}(\lambda_1 + \cdots + \lambda_n) = \frac{1}{n}tr(A)$