MA 3113 Practice Exam 2 Solutions

1.) Let $f, g \in C'(\mathbb{R}) \Rightarrow f', g'$ both exist and are cont. By properties of $\frac{d}{dx}$, $(f+g)'(x) = f'(x) + g'(x)$ $\therefore (f+g)'$ exists by continuity $f'(x) + g'(x)$ is cont. $\therefore f+g \in C'(\mathbb{R})$. Let $c \in \mathbb{R}$ and again $(cf)'(x) = cf'(x)$ so $(cf)'$ exists and by cont. $(cf)'$ is cont. $\therefore cf \in C'(\mathbb{R})$. Finally $\frac{d}{dx}(0) = 0$ and is cont. $\therefore 0 \in C'(\mathbb{R})$. So $C'(\mathbb{R})$ is subsp of $C(\mathbb{R})$.

2.) Let $f, g \in C_a[a,b]$. then consider $(f+g)(a) = f(a) + g(a) = 0 + 0 = 0$ since $f, g \in C_a[a,b]$. $\therefore f+g \in C_a[a,b]$. Let $c \in \mathbb{R}$, consider $(cf)(a) = cf(a) = c \cdot 0 = 0$ $\therefore cf \in C_a[a,b]$. clearly $0 \in C_a[a,b]$.

3.) Notice $\dim S = 3$ and $\dim P_2 = 3$. $\therefore$ just need to check if S is lin. indep. consider $c_1(t^2+1) + c_2(t^2+t) + c_3(t+1) = 0$

$\Rightarrow (c_1+c_2)t^2 + (c_2+c_3)t + (c_1+c_3)\cdot 1 = 0 \Rightarrow \begin{cases} c_1+c_2=0 \\ c_2+c_3=0 \\ c_1+c_3=0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1&1&0&0\\0&1&1&0\\1&0&1&0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1&0&0&0\\0&1&0&0\\0&0&1&0 \end{array}\right) \Rightarrow c_1 = c_2 = c_3 = 0$.

$\therefore$ S is lin indep, thus a basis for P_2

4.) Notice $\dim S = \dim P_2 = 3$. $\therefore$ just need to check if S is lin. indep. Consider $c_1(2t^2+1) + c_2(3t^2-4t) + c_3(4t+9) = 0$

$\Rightarrow (2c_1+3c_2)t^2 + (-4c_2+4c_3)t + (c_1+9c_3)\cdot 1 = 0 \Rightarrow \begin{cases} 2c_1+3c_2=0 \\ -4c_2+4c_3=0 \\ c_1+9c_3=0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 2&3&0&0\\0&-4&4&0\\1&0&9&0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1&0&9&0\\0&1&-1&0\\0&0&0&0 \end{array}\right)$

$\Rightarrow$ get a free variable. $\therefore$ S is ~~not~~ lin. dep. $\therefore$ not a basis for P_2

5.) $T(p) = \int_0^1 (a_0 + a_1 t + a_2 t^2)\,dt = \left(a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3}\right)\Big|_0^1 = a_0 + \frac{a_1}{2} + \frac{a_2}{3}$. know $Ker(T) = \{p \in P_2 : T(p) = 0\}$.

$\therefore T(p) = 0 \Rightarrow a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0 \Rightarrow a_0 = -\frac{1}{2}a_1 - \frac{1}{3}a_2 \Rightarrow p(t) = -\frac{1}{2}a_1 - \frac{1}{3}a_2 + a_1 t + a_2 t^2 = a_1(t - \frac{1}{2}) + a_2(t^2 - \frac{1}{3})$

So $Ker(T) = \text{span}\{t - \frac{1}{2}, t^2 - \frac{1}{3}\}$

6.) $T(p) = \frac{d}{dt}(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$. know $Ker(T) = \{p \in P_2 : T(p) = 0\}$. $\therefore$ $T(p) = 0 \Rightarrow a_1 + 2a_2 t = 0$ for all t

$\Rightarrow a_1 = a_2 = 0$. $\therefore p(t) = a_0$ $\therefore Ker(T) = \text{span}\{1\}$

7.) Notice $d^2 - 9d = 0 \Rightarrow d(d-9) = 0$ $\therefore d = 0$ or $d = 9$. If $d = 9 \Rightarrow \dim Ker(A) = 9$. so by Rank-null Thm

$\Rightarrow rk(A) = 10 - 9 = 1$. so $\dim Col(A) = 1$. If $d = 0 \Rightarrow \dim Ker(A) = 0$, so by Rank-null Thm $\Rightarrow rk(A) = 10 - 0 = 10$

$\therefore \dim Col(A) = 10$. A is invertible in the case $d = 0$.

8.) Consider eq. $Ax = 0$. $\Rightarrow A^t(Ax) = A^t 0 \Rightarrow (A^t A)x = 0$. Since $A^t A$ is nonsingular $\Rightarrow x = 0$ is only sol'n.

$\therefore$ $Ax = 0$ only has sol'n $x = 0$. $\Rightarrow \dim Ker(A) = 0$. so by Rank-null Thm, $rk(A) = n - 0 = n$.

9.) Consider $\det(A) = \begin{vmatrix} 2&6&0\\1&3&2\\3&9&2 \end{vmatrix} = 2\begin{vmatrix} 3&2\\9&2 \end{vmatrix} - 6\begin{vmatrix} 1&2\\3&2 \end{vmatrix} = 2(6-18) - 6(2-6) = 2(-12) - 6(-4)$

$= -24 + 24 = 0$.

Since $\det(A) = 0 \Rightarrow A$ not invertible

10.) consider $\det(A) = \begin{vmatrix} 2 & 2 & 2 \\ 4 & 4 & 4 \\ 8 & 8 & 8 \end{vmatrix} \overset{\text{via rref}}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$. $\therefore$ A is not invertible

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11.) S'pose A is invertible, then $\det(A) \neq 0$. OTOH, $\det(A^3) = 0 \Rightarrow (\det(A))^3 = 0 \Rightarrow \det(A) = 0$ contradiction.
$\therefore$ A is not invertible

12.) consider $\det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(P^{-1})\det(A) = \det(PP^{-1})\det(A) = \det(A)$

13.) Consider $Ax = x \Rightarrow Ax - x = 0 \Rightarrow (A - I_n)x = 0$. Since no nonzero x solves $Ax = x \Rightarrow x = 0$ only soln.
$\therefore$ $x = 0$ is only soln to $(A - I_n)x = 0$. $\therefore$ $\dim \ker(A - I_n) = 0$. $\therefore$ by Rank-Null Thm $\Rightarrow$ rk$(A - I_n) = n$
$\therefore$ $A - I_n$ nonsingular.