MA 3113 Practice Final Exam Solutions

1.) $p_A(\lambda) = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)+2 = 3-3\lambda-\lambda+\lambda^2+2 = \lambda^2-4\lambda+5$, $p_A(\lambda)=0 \Rightarrow \lambda = \frac{4\pm\sqrt{16-20}}{2} = 2\pm 2i$

$\lambda = 2-2i$: $(A-(2-2i)I_2)v=0 \Rightarrow \left(\begin{array}{cc|c} -1+2i & -2 & 0 \\ 1 & 1+2i & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 1+2i & 0 \\ 0 & 0 & 0 \end{array}\right)$ so $v = \begin{pmatrix} \cancel{1+2i} \\ -1 \end{pmatrix} v_2$, pick $v_2=1 \Rightarrow v = \begin{pmatrix} 1+2i \\ -1 \end{pmatrix}$

for $\lambda = 2+2i$: $v = \overline{\begin{pmatrix} 1+2i \\ -1 \end{pmatrix}} = \begin{pmatrix} 1-2i \\ -1 \end{pmatrix}$

2.) $p_A(\lambda) = \begin{vmatrix} 3-\lambda & 1 \\ -2 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda)+2 = 15-8\lambda+\lambda^2+2 = \lambda^2-8\lambda+17$, $p_A(\lambda)=0 \Rightarrow \lambda = \frac{8\pm\sqrt{64-68}}{2} = 4\pm 2i$

$\lambda = 4-2i$: $(A-(4-2i)I_2)v=0 \Rightarrow \left(\begin{array}{cc|c} -1+2i & 1 & 0 \\ -2 & 1+2i & 0 \end{array}\right) \xrightarrow{\text{ref}} \left(\begin{array}{cc|c} -2 & 1+2i & 0 \\ 0 & 0 & 0 \end{array}\right)$ so $v = \begin{pmatrix} 1+2i \\ 2 \end{pmatrix} v_2$ pick $v_2=1$ so $v = \begin{pmatrix} 1+2i \\ 2 \end{pmatrix}$

for $\lambda = 4+2i$: $v = \overline{\begin{pmatrix} 1+2i \\ 2 \end{pmatrix}} = \begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$

3.) Consider $Px=\lambda x$ for $x\neq 0$. Then $\lambda x = Px = P^2x = P(Px) = P(\lambda x) = \lambda Px = \lambda^2 x$. $\therefore (\lambda-\lambda^2)x=0$ and $x\neq 0$ $\therefore \lambda-\lambda^2=0 \Rightarrow \lambda = 0$ or 1 are possible.

4.) Consider $Ax=\lambda x$ for $x\neq 0$. Then $A^2x=\lambda Ax = \lambda^2 x$. but $A^2=0 \Rightarrow \lambda^2 x = 0$. But $x\neq 0 \therefore \lambda^2 = 0 \Rightarrow \lambda = 0$

5.) Know have $Ax=\lambda x$ for $x\neq 0$. Then $A^2x = \lambda Ax = \lambda^2 x \Rightarrow A^3x = \lambda^2 Ax = \lambda^3 x$, proceeding in this fashion get $A^k x = \lambda^k x$ for some $k\geq 1$.

6.) By def. of eigenvalues $(A-\lambda I_n)x=0$ has non trivial soln, $x\neq 0$. if $\lambda=0$ is an eigenvalue to A $\Rightarrow Ax=0$ has nontrivial soln i.e. $rk(A)\neq n$. $\therefore$ A is not invertible.

7.) set $v_1=1, v_2=t, v_3=t^2$. Set $w_1=1$ notice $\|w_1\|^2=\int_0^1 dt = 1$. so $w_2 = v_2 - \text{Proj}_{w_1}(v_2) = t - \frac{\langle t,1\rangle}{\|w_1\|^2}\cdot 1$ but $\langle t,1\rangle = \int_0^1 t\,dt = \frac{1}{2}$. $\therefore w_2 = t-\frac{1}{2}$. So $w_3 = v_3 - \text{Proj}_{w_1}(v_3) - \text{Proj}_{w_2}(v_3) = t^2 - \frac{\langle t^2,1\rangle}{\|w_1\|^2}\cdot 1 - \frac{\langle t^2, t-\frac{1}{2}\rangle}{\|w_2\|^2}\cdot(t-\frac{1}{2})$ then $\langle t^2,1\rangle = \int_0^1 t^2 dt = \frac{1}{3}$, $\langle t^2, t-\frac{1}{2}\rangle = \int_0^1 (t^3-\frac{1}{2}t^2)dt = \frac{1}{12}$. $\|t-\frac{1}{2}\|^2 = \int_0^1 (t-\frac{1}{2})^2 dt = \frac{1}{12}$

$\therefore w_3 = t^2-(t-\frac{1}{2})-\frac{1}{3} = t^2-t-\frac{1}{6}$, then $\|w_3\|^2 = \int_0^1 (t^2-t-\frac{1}{6})^2 dt = \frac{7}{60}$. $\therefore u_1 = 1$, $u_2 = \sqrt{12}(t-\frac{1}{2})$, $u_3 = 2\sqrt{\frac{15}{7}}(t^2-t-\frac{1}{6})$

8.) same as 7 but stop after w_2 to get $u_1=1$, $u_2=\sqrt{12}(t-\frac{1}{2})$

9.) Since P has orthonormal columns $\Rightarrow P^tP=I$. $\therefore \|Px\|^2 = \langle Px,Px\rangle = \langle x, P^tPx\rangle = \langle x, Ix\rangle = \langle x,x\rangle = \|x\|^2$ $\Rightarrow \|Px\| = \|x\|$.

10.) Since P has orthonormal columns $\Rightarrow P^tP=I$. $\therefore \langle Px,Py\rangle = \langle x, P^tPy\rangle = \langle x, Iy\rangle = \langle x,y\rangle$ (*)

$\therefore$ if $\langle Px,Py\rangle = 0$, (*) $\Rightarrow \langle x,y\rangle$. OTOH if $\langle x,y\rangle = 0$, (*) $\Rightarrow \langle Px,Py\rangle = 0$

11.) let λ be an eigenvalue for A. Then $\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Av, v\rangle = \langle v, A^t v\rangle = \langle v, \bar{A}^t v\rangle$ as A real

$= \langle v, \bar{\lambda} v\rangle = \bar{\lambda}\langle v, v\rangle$. $\therefore$ $\lambda \|v\|^2 = \bar{\lambda}\|v\|^2$. Since $v \neq 0$ as $Av = \lambda v$ $\Rightarrow$ $\lambda = \bar{\lambda}$.

12.) let λ_1, λ_2 be two different eigenvalues. $\therefore$ there are $v_1, v_2 \neq 0$ s.t. $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$

consider $\lambda_1\langle v_1, v_2\rangle = \langle \lambda_1 v_1, v_2\rangle = \langle Av_1, v_2\rangle = \langle v_1, A^t v_2\rangle = \langle v_1, Av_2\rangle = \langle v_1, \lambda_2 v_2\rangle = \lambda_2\langle v_1, v_2\rangle$

$\Rightarrow (\lambda_1 - \lambda_2)\langle v_1, v_2\rangle = 0$. Since $\lambda_1 \neq \lambda_2 \Rightarrow \langle v_1, v_2\rangle = 0$.