Name:______________________________

MA 3163 Section 01 *Practice Exam 1* *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. Let G be an Abelian group and define

$$H = \{x \in G : x^n = e \text{ for some odd integer } n\}$$

Prove that $H \leq G$. (HINT: This n may change for different x's)

2. Let $n \geq 3$ and define

$$H = \{\alpha \in S_n : \alpha(1) = 1 \text{ or } 2 \text{ and } \alpha(2) = 1 \text{ or } 2\}$$

Prove that $H \leq G$. Determine $|H|$.

3. Let G be a cyclic group that has exactly three distinct subgroups: G itself, $\{e\}$, and a subgroup of order 5. Prove that $G \cong Z_{25}$.

4. Find an example of a non-Abelian group whose proper subgroups are all cyclic.

5. Let p be prime and let G be a non-Abelian group such that $|G| = p^3$ and $Z(G) \neq \{e\}$. Prove that $Z(G)$ is cyclic.

6. If G is a group and $|G : Z(G)| = 4$, prove that $G/Z(G) \cong Z_2 \oplus Z_2$.

7. Let $\mathbb{C}^\times$ be the set of nonzero complex numbers under multiplication and $\mathbb{R}_{\geq 0}$ the set of all positive real numbers under multiplication. It is known that $\mathbb{C}^\times$ and $\mathbb{R}_{\geq 0}$ are both groups. Let $H = \{z \in \mathbb{C}^\times : |z| = 1\}$. Prove that $\mathbb{C}^\times/H \cong \mathbb{R}_{\geq 0}$.

8. Prove that every group homomorphism $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ has the form $\varphi(x, y) = ax + by$ for integers a and b. Describe the kernel of φ.

9. Let G be a group and let $g \in G$. If $z \in Z(G)$, show the inner automorphism induced by g is the same as the inner automorphism induced by zg.