Practice Exam 1 Solutions

1.) let G be Abelian. $H = \{x \in G : x^n = e$ for some odd integer $n\}$. Prove $H \leq G$.

pf:
Consider $x, y \in H$, then $\exists\, n_1, n_2$ odd integers s.t. $x^{n_1} = e$ and $y^{n_2} = e$.
Then consider $(xy)^{n_1 n_2} = (x^{n_1})^{n_2}(y^{n_2})^{n_1} = e$ as G Abelian but is $n_1 n_2$ odd?
let $n_1 = 2k+1$ and $n_2 = 2l+1$ for $k, l \in \mathbb{Z}$ then $n_1 n_2 = (2k+1)(2l+1) = 2(2kl + l + k) + 1$ odd
so $xy \in H$. Then consider $(x^{-1})^{n_1} = (x^{n_1})^{-1} = e^{-1} = e$ and n_1 is odd. So $H \leq G$ □

2.) let $n \geq 3$, $H = \{\alpha \in S_n : \alpha(1) = 1$ or 2 and $\alpha(2) = 1$ or $2\}$. Prove $H \leq G$ and compute $|H|$

pf:
let $\alpha, \beta \in H$. then $(\alpha\beta)(1) = \alpha(\beta(1)) = \begin{cases} \alpha(1) \\ \alpha(2) \end{cases} = \begin{cases} 1 \\ 2 \\ 1 \\ 2 \end{cases} = \begin{cases} 1 \\ 2 \end{cases}$ and $(\alpha\beta)(2) = \alpha(\beta(2)) = \begin{cases} \alpha(1) \\ \alpha(2) \end{cases} = \begin{cases} 1 \\ 2 \end{cases}$
so $\alpha\beta \in H$. consider $\alpha^{-1}(1)$. since $\alpha \in S_n$ α^{-1} exists so $\alpha(1) = 1 \Rightarrow \alpha^{-1}(1) = 1$
if $\alpha(1) = 2 \Rightarrow \alpha^{-1}(2) = 1$ similarly if $\alpha(2) = 1 \Rightarrow \alpha^{-1}(1) = 2$ and if $\alpha(2) = 2 \Rightarrow \alpha^{-1}(2) = 2$
so $\alpha^{-1} \in H$. So $H \leq G$. Since we are choosing where to permute 1 and 2 gives $(n-2)!$ choices
but we do this twice so $|H| = 2(n-2)!$. □

3.) Let G be cyclic s.t. G, $\{e\}$ and H s.t. $|H| = 5$ are the only subgrps. Prove $G \cong \mathbb{Z}_{25}$

pf:
let $|G| = n$ since G cyclic all the divisors are the orders of all its subgrps. so 5 and $\frac{n}{5}$
are divisors of n if $\frac{n}{5} \neq 5 \Rightarrow n$ has at least 4 divisors: $1, 5, 25$ and something else
but there are exactly 3 subgrps so $\frac{n}{5} = 5 \Rightarrow n = 25$ so $G \cong \mathbb{Z}_{25}$ □

4.) Find an example of a non-Abelian grp whose proper subgrps are all cyclic.

pf:
$S_3 = \{(1), (12), (13), (23), (123), (132)\}$ works. let $H_1 = \{(1), (12)\}$, $H_2 = \{(1), (13)\}$
$H_3 = \{(1), (23)\}$, and $H_4 = \{(1), (123), (132)\} = A_3$ One can check these
are all the proper subgrps but $H_1 \cong H_2 \cong H_3 \cong \mathbb{Z}_2$ as each have a generator of order 2
and $H_4 \cong \mathbb{Z}_3$ as both nontrivial elements have order 3 and generate H_4. □

5.) let p be prime and G non-Abelian grp s.t. $|G| = p^3$ and $Z(G) \neq \{e\}$. Prove $Z(G)$ cyclic.

pf:
Since $Z(G) \leq G$, then by Lagrange's Thm $|Z(G)| \mid |G| \Rightarrow |Z(G)| = 1, p, p^2$ or p^3
$|Z(G)| \neq 1$ as $Z(G) \neq \{e\}$ and $|Z(G)| \neq p^3$ as $\Rightarrow Z(G) = G$ but G non-Abelian. so
If $|Z(G)| = p^2 \Rightarrow |G/Z(G)| = \frac{|G|}{|Z(G)|} = \frac{p^3}{p^2} = p \Rightarrow G/Z(G)$ is cyclic $\Rightarrow G$ Abelian
contradiction. so only possibility is $|Z(G)| = p \Rightarrow Z(G)$ is cyclic □

6.) If G is a grp and $|G : Z(G)| = 4$, Prove $G/Z(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

pf:
Since $|G : Z(G)| = 4 \Rightarrow |G/Z(G)| = 4 = 2^2$ so $\Rightarrow G/Z(G) \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

if $G/Z(G) \cong \mathbb{Z}_4 \Rightarrow G/Z(G)$ cyclic $\Rightarrow G$ Abelian so $G = Z(G) \Rightarrow |G/Z(G)| = 1$

$\Rightarrow |G : Z(G)| = 1 \neq 4$ so $G/Z(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\square$

7.) let $(\mathbb{C}^\times = \{ z \in \mathbb{C} : z \neq 0 \}, \cdot)$, $(\mathbb{R}_{\geq 0}, \cdot)$ be grps let $H = \{ z \in \mathbb{C}^\times : |z| = 1 \}$
Prove ~~$\mathbb{C}^\times/\mathbb{R}_{\geq 0}$~~ $\mathbb{C}^\times/H \cong \mathbb{R}_{\geq 0}$.

pf:
Define $\varphi : \mathbb{C}^\times \to \mathbb{R}_{\geq 0}$ via $\varphi(z) = |z|$, then φ is a grp homomorphism as $\varphi(z_1 z_2) = |z_1 z_2| = |z_1||z_2| = \varphi(z_1)\varphi(z_2)$. Then $\ker\varphi = \{ z \in \mathbb{C}^\times : \varphi(z) = 1 \}$ but $\varphi(z) = |z|$ so $\ker\varphi \cong H$. So by 1st iso thm $\Rightarrow \mathbb{C}^\times/\ker\varphi \cong \varphi(\mathbb{C}^\times)$ but $\varphi(\mathbb{C}^\times) = \mathbb{R}_{\geq 0}$ as $\varphi(z)$ is onto since given $y \in \mathbb{R}_{\geq 0}$ can always find $z \in \mathbb{C}$ s.t. $|z| = y$ so $\mathbb{C}^\times/H \cong \mathbb{R}_{\geq 0}$ $\square$

8.) Prove that every grp homomorphism $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ has form $\varphi(x, y) = ax + by$ for $a, b \in \mathbb{Z}$. Describe $\ker\varphi$.

pf:
1st note since φ is homom. $\Rightarrow \varphi((x_1, y_1) + (x_2, y_2)) = \varphi(x_1, y_1) + \varphi(x_2, y_2)$.
Then consider $\varphi(0,1) = b$ and $\varphi(1,0) = a$ for some $a, b \in \mathbb{Z}$. then
$\varphi(x, 0) = \varphi(\underbrace{(1,0) + \cdots + (1,0)}_{x \text{ times}}) = \underbrace{\varphi(1,0) + \cdots + \varphi(1,0)}_{x \text{ times}} = ax$ similarly $\varphi(0, y) = by$
then finally, $(x, y) = (x, 0) + (0, y)$ so $\varphi(x, y) = \varphi((x,0) + (0,y)) = \varphi(x,0) + \varphi(0,y) = ax + by$
the $\ker\varphi = \{ (x,y) : ax + by = 0 \}$ gives a dotted line through the origin. $\square$

9.) let G be grp and $g \in G$. if $z \in Z(G)$ show $\varphi_g = \varphi_{zg}$.

pf:
consider $\varphi_{zg}(x) = zgx(zg)^{-1} = zgxg^{-1}z^{-1} = zz^{-1}gxg^{-1} = gxg^{-1} = \varphi_g$.
Since $z \in Z(G)$. $\square$