MA 3163 Practice Exam 2 Solutions

1.) let G be an Abelian grp s.t. $|G| = 16$. Suppose $\exists\, a, b \in G$ s.t. $|a| = |b| = 4$ and $a^2 \neq b^2$. Determine the isomorphism class of G.

pf: 1st since $|G| = 2^4$, by Fundamental thm of finite Abelian grps $G \cong \mathbb{Z}_{16}$, $G \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2$, $G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$, $G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, or $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. 1st $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has no elements of order 4. Now since in $\mathbb{Z}_n$, a^2 is really $2a$. so in $\mathbb{Z}_{16}$ $|4| = |12| = 4$ but $2 \cdot 4 = 8$ and $2 \cdot 12 = 24 \equiv_{16} 8$ so $2(4) = 2(12)$ in $\mathbb{Z}_{16}$ so $G \ncong \mathbb{Z}_{16}$
in $\mathbb{Z}_8 \oplus \mathbb{Z}_2$, $|(2,1)| = |(6,1)| = 4$ but $2(2,1) = (4,1) = 2(6,1)$ so $G \ncong \mathbb{Z}_8 \oplus \mathbb{Z}_2$
finally in $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $|(1,1,1)| = |(3,1,1)| = 4$ but $2(1,1,1) = (2,2,2) = 2(3,1,1)$ so $G \ncong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
Thus $G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$ □

2.) let p, q, r be distinct primes. let G be Abelian s.t. $|G| = pqr$. What is the possible isos for G.

pf: by Fundamental thm of finite Abelian grps $G \cong \mathbb{Z}_{pqr}$, $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_{qr}$, $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Z}_r$ or $G \cong \mathbb{Z}_{pq} \oplus \mathbb{Z}_r$. but a lemma said $\mathbb{Z}_p \oplus \mathbb{Z}_q \cong \mathbb{Z}_{pq}$ iff $\gcd(p,q) = 1$
$\therefore$ since p, q, r distinct primes $G \cong \mathbb{Z}_{pqr}$ or any of them. □

3.) let R be comm ring w/unity. A, B ideals of R s.t. $R = A + B$. Prove $A \cap B = AB$.

pf: let $x \in A \cap B$. notice let 1 be unity of R so $1 = a + b$ for $a \in A$ and $b \in B$ then $x = xa + xb$ since A, B ideals $xa \in A$ and $xb \in B$ OTH. $x \in A \cap B \Rightarrow x \in A$ and $x \in B$ so $xa \in B$ and $xb \in A$ as A, B ideals but by def. of $AB \Rightarrow xa + xb \in AB$ so $x \in AB$ □

4.) Prove that $I = \langle 2, x, y \rangle$ is maximal in $\mathbb{Z}[x,y]$.

pf: 1st notice $\mathbb{Z}[x,y]$ is a comm. ring w/ unity. the unity is the 1 polynomial and since $\mathbb{Z}$ comm. so $\mathbb{Z}[x,y]$ comm. next consider the ideal $I_1 = \langle x, y \rangle$ then $I_1 = \{ax + by : a, b \in \mathbb{Z}\}$ so $\mathbb{Z}[x,y]/I_1 = \{c + I_1 : c \in \mathbb{Z}\}$ then $\mathbb{Z}[x,y]/I = \{c + I : c \in \mathbb{Z}_2\} = \{0 + I, 1 + I\}$. Claim: $\mathbb{Z}[x,y]/I$ is a field consider $\varphi : \mathbb{Z}[x,y] \to \mathbb{Z}_2$ via $\varphi(c + I) = c$ then $\ker\varphi = I$ so by 1st isom. thm $\mathbb{Z}[x,y]/I = \mathbb{Z}[x,y]/\ker\varphi \cong \varphi(\mathbb{Z}[x,y]) = \mathbb{Z}_2$ so $\mathbb{Z}[x,y]/I$ is a field $\Rightarrow I$ is maximal. □

5.) let R be a ring w/unity, $a \in R$ unit. Prove $\varphi(x) = axa^{-1}$ is ring homom.

pf: consider $\varphi(x+y) = a(x+y)a^{-1} = axa^{-1} + aya^{-1} = \varphi(x) + \varphi(y)$. then let 1 be unity so $\varphi(xy) = axya^{-1} = ax1ya^{-1} = (axa^{-1})(aya^{-1}) = \varphi(x)\varphi(y)$. □

6.) let $R = \{ A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \}$ Prove $\varphi(A) = a-b$ ring homom. Compute $\ker\varphi$.

pf: let $A_1, A_2 \in R$ then $A_1 + A_2 = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & a_1+a_2 \end{pmatrix}$, $A_1A_2 = \begin{pmatrix} a_1a_2+b_1b_2 & a_1b_2+b_1a_2 \\ b_1a_2+a_1b_2 & b_1b_2+a_1a_2 \end{pmatrix}$

Consider $\varphi(A_1+A_2) = (a_1+a_2) - (b_1+b_2) = (a_1-b_1) + (a_2-b_2) = \varphi(A_1)+\varphi(A_2)$

and $\varphi(A_1A_2) = (a_1a_2+b_1b_2) - (a_1b_2+b_1a_2) = a_1a_2 - a_1b_2 + b_1b_2 - b_1a_2$

$= a_1(a_2-b_2) + b_1(b_2-a_2) = (a_1-b_1)(a_2-b_2) = \varphi(A_1)\varphi(A_2)$.

then $\ker\varphi = \{ A \in R : \varphi(A) = 0 \}$ $\varphi(A) = 0 \Rightarrow a = b$ so $\ker\varphi = \{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \}$. ∎

7.) construct a field w/ 27 elements.

pf: consider $p(x) = x^3 + 2x^2 + 1$ over $\mathbb{Z}_3$. then $p(0) = 1$, $p(1) = 1$, $p(2) = 2$ so $p(x)$ irreduc. over $\mathbb{Z}_3$

$\Rightarrow \langle p(x) \rangle$ is a maximal ideal in $\mathbb{Z}_3[x]$ as $\mathbb{Z}_3$ is a field. so by div. algo.

if $f(x) \in \mathbb{Z}_3[x]/\langle p(x) \rangle \Rightarrow f(x) = p(x)q(x) + r(x)$ w/ $\deg r(x) < \deg p(x)$

$\Rightarrow r(x) = ax^2 + bx + c \quad \therefore \quad \mathbb{Z}_3[x]/\langle p(x) \rangle = \{ ax^2+bx+c : a, b, c \in \mathbb{Z}_3 \} \Rightarrow$ # elmts $3^3 = 27$ ∎

8.) let $f(x) \in \mathbb{Z}_m[x]$. what criteria is needed on $f(x)$ and m s.t. $\mathbb{Z}_m[x]/\langle f(x) \rangle$ is a field w/ m^n elements.

pf: 1st need $f(x)$ to be irreducible over a field F, then $F[x]/\langle f(x) \rangle$ will be a field as $\langle f(x) \rangle$ will be maximal. Then $\mathbb{Z}_m$ is a field iff m is prime. Finally to get m^n elements, need $\deg f(x) = n$ as $\mathbb{Z}_m[x]/\langle f(x) \rangle$ will consist of polys of deg $n-1$ or less, meaning n coefficients w/ m choices so m^n elmts ∎

9.) Prove that $I = \langle 2+i \rangle$ is maximal in $\mathbb{Z}[i]$

pf: let $R = \mathbb{Z}[i]/I$. then $2+i+I = 0+I$ so in R $2+i = 0 \Rightarrow i = -2$

$\Rightarrow -1 = 4 \Rightarrow 0 = 5$ in R so $R = \{ a + -2b + I : a, b \in \mathbb{Z} \} = \{ c + I : c = 0, 1, 2, 3, 4 \}$

next define $\varphi : R \to \mathbb{Z}_5$ via $\varphi(c+I) = c$, natural homo. so $\ker\varphi = I$

and $\varphi(R) = \mathbb{Z}_5$ so by 1st iso thm $R \cong \mathbb{Z}_5$. $\Rightarrow R$ is a field but $\mathbb{Z}[i]$ is a comm. ring w/ unity so I is maximal. ∎