# Practice Exam 1 Solutions

1.) let $g \in G$ s.t. $|g| = 2$. set $K = Z(G) \cup gZ(G)$.

pf:

let $x, y \in K$, want to show $xy^{-1} \in K$. There are 4 cases: (1) $x, y \in Z(G)$, (2) $x, y \in gZ(G)$ (3) $x \in Z(G)$, $y \in gZ(G)$, (4) $x \in gZ(G)$, $y \in Z(G)$.

(1) $x, y \in Z(G)$ done since $Z(G)$ subgroup of $G \Rightarrow xy^{-1} \in Z(G)$ so $xy^{-1} \in K$.

(2) $x, y \in gZ(G)$ then $x = gz_1$ and $y = gz_2$ so $xy^{-1} = gz_1(gz_2)^{-1} = gz_1z_2^{-1}g = g^2 z_1 z_2^{-1} = z_1 z_2^{-1} \in Z(G)$ since $g$ order 2 and $z_1, z_2 \in Z(G)$ so they commute w/ everything in $G$. So $xy^{-1} \in K$.

(3) $x \in Z(G)$, $y \in gZ(G) \Rightarrow y = gz_1$ so $xy^{-1} = x(gz_1)^{-1} = xz_1^{-1}g^{-1} = g^{-1}xz_1^{-1}$ since $x, z_1 \in Z(G)$ but $g^{-1}xz_1^{-1} = gxz_1^{-1} \in gZ(G)$ since $g$ has order 2. $\Rightarrow xy^{-1} \in K$

(4) $x \in gZ(G)$, $y \in Z(G) \Rightarrow x = gz_1$, so $xy^{-1} = gz_1y^{-1} \in gZ(G)$ since $Z(G) \leq G$. $\Rightarrow xy^{-1} \in K$

So $xy^{-1} \in K$ in all cases $\Rightarrow K \leq G$. □

2.) let $H \leq G$, define $N(H) = \{x \in G : x^{-1}Hx = H\}$. Prove $N(H) \leq G$.

pf:

pick $x, y \in N(H)$ and consider $(xy)^{-1}H(xy) = (y^{-1}x^{-1})H(xy) = y^{-1}(x^{-1}Hx)y = y^{-1}Hy$ since $x \in N(H)$ so $y^{-1}Hy = H$ since $y \in N(H) \Rightarrow xy \in N(H)$

Now showing $x^{-1} \in N(H)$. Notice $x^{-1}Hx = H \iff xHx^{-1} = H$. (this is a claim)

→ since $x^{-1}Hx = H \Rightarrow \exists\, h_1, h_2$ s.t. $x^{-1}h_1x = h_2$ $\Rightarrow h_1 = xh_2x^{-1}$ so $\Rightarrow H \subseteq xHx^{-1}$ and $xHx^{-1} \subseteq H$ simultaneously

so if $x \in N(H) \Rightarrow x^{-1}Hx = H$ but $\Rightarrow xHx^{-1} = H \Rightarrow (x^{-1})^{-1}H(x^{-1}) = H$ or $x^{-1} \in N(H)$. □

3.) let $G$ be cyclic and $|G| = 2\cdot 5\cdot 7$. list all subgroups.

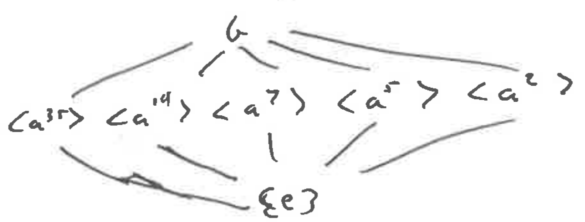
pf:

Since $G$ cyclic $\Rightarrow \exists\, a \in G$ s.t. $G = \langle a \rangle$. By Fundamental thm of cyclic groups every subgroup is cyclic and the divisors of $|G|$ are the order of each subgroup and the generators are $a^{\frac{|G|}{\text{divisor}}}$. So if $H \leq G$ then the orders are ~~$2, 2^2, 2^3, 5, 7, 2\cdot 5, 2\cdot 7, 5\cdot 7, 2^2\cdot 5, 2^2\cdot 7$~~, $2^2\cdot 5\cdot 7$

$|H| = 2, 5, 7, 2\cdot 5, 2\cdot 7, 5\cdot 7, 1$

so ~~$\langle a^{2^2\cdot 5\cdot 7}\rangle$~~ $\langle a^{2\cdot 7}\rangle$, order 5

here $\langle a^{35} \rangle$ w/ order 2, $\langle a^5 \rangle$ w/ order 14, $\langle a^{14} \rangle$ w/ order 5, $\langle a^2 \rangle$ order 35, $\langle a^7 \rangle$ w/ order 10, $\{e\}$ order 1




□

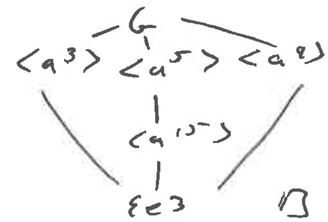
4.) let $G$ be cyclic and $|G| = 3^2 \cdot 5$. list all subgrps.

pf: since $G$ cyclic $\Rightarrow \exists\, a \in G$ s.t. $G = \langle a \rangle$, by Fund thm of cyclic grps, every subgrp is cyclic and if $k$ is a divisor of $G$ then orders of all the subgrps are these divisors and the generators are $\langle a^{\frac{|G|}{k}} \rangle$. so if $H \leq G$ then the possible orders are $|H| = 1, 3, 3^2, 5, 3\cdot 5$

so have $\langle a^{15} \rangle$ w/ order 3   $\langle a^3 \rangle$ w/ order 15

$\langle a^5 \rangle$ w/ order 9   $\{e\}$ w/ order 1

$\langle a^9 \rangle$ w/ order 5




□

5.) let $G = S_3$ and $H = \{(1), (12)\}$, compute left cosets of $H$ in $G$.

pf: 1st $S_3 = \{(1), (12), (13), (23), (123), (132)\}$ recall $aH = \{ah : h \in H\}$ is a coset.

$(1)H = H$, $(12)H = \{(12), (12)(12)\} = \{(12), (1)\} = H$, $(13)H = \{(13), (13)(12)\}$
$= \{(13), (123)\} = (123)H$

$(23)H = \{(23), (23)(12)\} = \{(23), (132)\} = (132)H$

notice $|G| = 6$ and $|H| = 2$ so # of cosets is 3 : $H$, ~~$\{(12), (12)\}$~~, $\{(13), (123)\}$, $\{(23), (132)\}$ □

6.) let $G = \mathbb{Z}_9$, $H = \{0, 3, 6\}$. compute left cosets of $H$ in $G$.

pf: 1st notice $|G| = 9$ and $|H| = 3$ so the # of cosets is 3. and $\mathbb{Z}_9 = \{0,1,2,3,4,5,6,7,8\}$

so $0 + H = H$, $1 + H = \{1, 4, 7\}$, $2 + H = \{2, 5, 8\}$, $3 + H = \{3, 6, 0\} = H$
$4 + H = \{1, 4, 7\}$, $5 + H = \{2, 5, 8\}$, $6 + H = \{0, 3, 6\}$, $7 + H = \{1, 4, 7\}$, $8 + H = \{2, 5, 8\}$

so they are $\{0, 3, 6\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$ □.

7.) let $G$ grp, $\varphi: G \to G$ automorphism, $H = \{g \in G : \varphi^2(g) = g\}$ show $H \leq G$.

pf: let $x, y \in H$ then consider $\varphi^2(xy^{-1}) = \varphi(\varphi(xy^{-1})) = \varphi(\varphi(x)\varphi(y^{-1}))$
$= \varphi^2(x)\varphi^2(y^{-1}) = xy^{-1}$ since $\varphi$ is automorphism so isom. so oper. preserv and by def of $\varphi$. so $xy^{-1} \in H$. $\Rightarrow H \leq G$. □

8.) let $G$ be a grp, $g, h \in G$. s'pose $g, h$ induce the same inner automorphism of $G$. show $h^{-1}g \in Z(G)$.

pf:

recall $\varphi_g(x) = gxg^{-1}$ $\forall x \in G$ is the inner automorphism for $g$ similarly $\varphi_h = hxh^{-1}$ $\forall x \in G$

so if $g, h$ induce the same $\Rightarrow$ $\varphi_g = \varphi_h$ $\forall x \in G$ so $gxg^{-1} = hxh^{-1}$ $\forall x \in G$

$\Rightarrow (h^{-1}g)x = x(h^{-1}g)$ $\Rightarrow$ $h^{-1}g$ commutes w/ every $x$ in $G$ $\Rightarrow$ $h^{-1}g \in Z(G)$ □

9.) let $G$ be a grp and $p$ a prime. s'pose $|G : Z(G)| = p^2$. Prove $G/Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$

pf:

since $|G : Z(G)| = p^2 \Rightarrow |G/Z(G)| = p^2$, know that $G/Z(G) \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \oplus \mathbb{Z}_p$

if $G/Z(G) \cong \mathbb{Z}_{p^2}$ $\Rightarrow$ $G/Z(G)$ is cyclic. But by the quotient center thm

$\Rightarrow G$ Abelian and $G = Z(G)$ but $\frac{|G|}{|Z(G)|} = p^2$ if $G = Z(G) \Rightarrow |G : Z(G)| = 1$ not $p^2$

so must have $G/Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ □