MA 4633/6833 Practice Exam 1 Solutions

1.) pf:

S.W.: want $\left|\frac{3n+2}{6n+1}-\frac{1}{2}\right|<\varepsilon \Rightarrow \left|\frac{2(3n+2)}{2(6n+1)}-\frac{6n+1}{2(6n+1)}\right| = \left|\frac{3}{2(6n+1)}\right|<\varepsilon \Rightarrow 6n+1>\frac{2}{3\varepsilon}$

$\Rightarrow n>\frac{1}{6}\left(\frac{2}{3\varepsilon}-1\right)$

Formal pf: $\forall \varepsilon>0$ pick N = next largest integer to $\frac{1}{6}\left(\frac{2}{3\varepsilon}-1\right)$. Then if $n\geq N$ consider

$\left|\frac{3n+2}{6n+1}-\frac{1}{2}\right| = \frac{3}{2(6n+1)} \leq \frac{3}{2(6N+1)}<\varepsilon$. $\square$

2.) pf:

S.W.: want $\left|\frac{n+2}{7n+5}-\frac{1}{7}\right|<\varepsilon \Rightarrow \left|\frac{7(n+2)}{7(7n+5)}-\frac{7n+5}{7(7n+5)}\right| = \left|\frac{9}{7(7n+5)}\right|<\varepsilon \Rightarrow 7n+5>\frac{7}{9\varepsilon}$

$\Rightarrow n>\frac{1}{7}\left(\frac{7}{9\varepsilon}-5\right)$

Formal pf: $\forall \varepsilon>0$, pick N = next largest integer to $\frac{1}{7}\left(\frac{7}{9\varepsilon}-5\right)$. Then if $n\geq N$ consider

$\left|\frac{n+2}{7n+5}-\frac{1}{7}\right| = \frac{9}{7(7n+5)} \leq \frac{9}{7(7N+5)}<\varepsilon$. $\square$.

3.) pf:

Since $\frac{a_n}{b_n}\to c$ and $c>0$, $\exists$ consts $m, M>0$ s.t. $m\leq \frac{a_n}{b_n}\leq M$ for $n\geq N$ $\Rightarrow mb_n\leq a_n\leq Mb_n$ for $n\geq N$. Next if $\sum_{n=1}^{\infty} b_n<\infty$ then $\sum_{n=1}^{\infty} Mb_n<\infty$ and if $\sum_{n=1}^{\infty} b_n$ diverges then so does $\sum_{n=1}^{\infty} mb_n$. $\therefore$ by the comparison theorem if $\sum_{n=1}^{\infty} b_n<\infty$ then $\sum_{n=1}^{\infty} a_n<\infty$ and if $\sum_{n=1}^{\infty} b_n$ diverges then so does $\sum_{n=1}^{\infty} a_n$. $\square$

4.) pf:

Since $\{b_n\}$ is a bdded seq. $\exists M>0$ s.t. $|b_n|\leq M$ $\forall n$: Consider $\sum_{n=1}^{\infty}|a_nb_n| \leq \sum_{n=1}^{\infty}|a_n||b_n|$ $\leq \sum_{n=1}^{\infty}|a_n|M = M\sum_{n=1}^{\infty}|a_n|<\infty$ since $\sum_{n=1}^{\infty} a_n$ converges absolutely. $\square$

5.) pf:

Since $A, B\subseteq \mathbb{R}$ are compact, by Heine-Borel $\Rightarrow$ A, B are both closed and bdded.
From a prop. in class $A\cup B$ is closed. Next let $x\in A\cup B \Rightarrow x\in A$ or $x\in B$.
Since A, B bdded $\exists$ consts. $M_1, M_2>0$ s.t. $|x|\leq M_1$ or $|x|\leq M_2$. Let $M=\max\{M_1, M_2\}$ so that $|x|\leq M$. This works $\forall x\in A\cup B$. $\therefore$ $A\cup B$ is bdded. Thus by Heine-Borel, $A\cup B$ compact. $\square$

6.) pf:

Since $A, B\subseteq\mathbb{R}$ are compact, by Heine-Borel $\Rightarrow$ A, B are both closed and bdded. From a prop. in class $A\cap B$ is closed. Next let $x\in A\cap B \Rightarrow x\in A$ and $x\in B$. Since A, B bdded, $\exists$ consts $M_1, M_2>0$ s.t. $|x|\leq M_1$ and $|x|\leq M_2$. Let $M=\max\{M_1, M_2\}$, then $|x|\leq M$. This works $\forall x\in A\cap B$ $\therefore$ $A\cap B$ is bdded. Hence by Heine-Borel, $A\cap B$ is compact. $\square$

7.) pf:

let $f: \mathbb{R} \to \mathbb{R}$ be cont. and $f(x) \in \mathbb{Q}$ $\forall x \in \mathbb{R}$. s'pose f is not const. Then $\exists\, x < y \in \mathbb{R}$ s.t. $f(x) < f(y)$ or $f(x) > f(y)$. wlog s'pose $f(x) < f(y)$. then since $f(x), f(y) \in \mathbb{Q}$, and $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists$ irrat'l # c s.t. $f(x) < c < f(y)$. Since f is cont on $[x, y]$ by IVT $\exists\, z \in (x, y)$ s.t. $f(z) = c$ but $f(z) \notin \mathbb{Q}$ contradiction. $\therefore$ f is const. $\square$

8.) pf:

Let $A \subseteq \mathbb{R}$ be closed and let $\{x_n\} \subseteq f^{-1}(A)$ s.t. $x_n \to x$. since $x_n \in f^{-1}(A)$ $\forall n \Rightarrow f(x_n) \in A$. Since f is cont $\Rightarrow f(x_n) \to f(x)$. and since A is closed $\Rightarrow f(x) \in A \Rightarrow x \in f^{-1}(A) \Rightarrow f^{-1}(A)$ closed. $\square$

9.) pf:

let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$. 1st notice if $x \neq 0$ then since $\sin x$ is cont. and pol'l fncs are cont. where they're defined, the limit laws $\Rightarrow$ $f(x)$ is cont. for $x \neq 0$. Just need to check continuity at $x = 0$. let $\{x_n\} \subseteq \mathbb{R}$ s.t. $x_n \to 0$, then consider $f(x_n) = x_n^2 \sin\left(\frac{1}{x_n}\right)$. Since fnc is bdd $\therefore$ $\forall n \geq 1$ $\exists\, M > 0$ s.t. $\left|\sin\left(\frac{1}{x_n}\right)\right| \leq M$ (in fact can choose $M = 1$ why?) and since $x_n \to 0$ as $n \to \infty$ $\Rightarrow x_n^2 \to 0$ as $n \to \infty$ thus $\left|x_n^2 \sin\left(\frac{1}{x_n}\right)\right| \leq x_n^2 M \to 0$ as $n \to \infty$ $\Rightarrow f(x_n) \to 0 = f(0)$. $\therefore$ f is cont. at $x = 0$. Thus f is cont. on $\mathbb{R}$ $\square$