Name:______________________________

MA 4633/6633 Section 01 — *Practice Exam 2* — *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. If f is differentiable at x_0 and $a \in \mathbb{R}$, prove that

$$\lim_{h\to 0} \frac{f(x_0 + ah) - f(x_0)}{h} = af'(x_0) \ .$$

2. Suppose g is continuous at $x = 0$. For what values of $n \in \mathbb{N}$ is the function $f(x) = x^n g(x)$ is differentiable at $x = 0$?

3. For $a, b > 0$ define $f(x)$ by

$$f(x) = \begin{cases} \displaystyle\int_a^b t^x \ dt & x \neq -1 \\ \log b - \log a & x = -1 \ . \end{cases}$$

Prove that f is continuous at $x = -1$.

4. Let $f(x) = x^2 \sin(\frac{1}{x})$ and $g(x) = \sin x$. Prove that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \quad \text{exists, but} \quad \lim_{x\to 0} \frac{f'(x)}{g'(x)} \quad \text{does not exist.}$$

5. Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$. If

$$\int_a^b f = 0 \ ,$$

prove that $f(x) = 0$ for all $x \in [a,b]$.

6. Suppose f is Lipschitz with constant L on $[0,1]$. Prove that

$$\left| \int_0^1 f(x)\, dx - \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{j}{n}\right) \right| \leq \frac{L}{n} \ .$$

7. Let $E \subseteq \mathbb{R}$. Recall χ_E is the function such that $\chi_E(x) = 1$ for $x \in E$ and zero otherwise. Does there exist a set $E \subseteq [0,1]$ such that for any $0 \leq a < b \leq 1$,

$$\int_a^b \chi_E = \frac{b-a}{2} \ ?$$

8. Let $f(x)$ be a continuous function on $[0,1]$ such that for every $0 \leq a < b \leq 1$ the following holds:

$$\int_a^{(a+b)/2} f(x)\, dx = \int_{(a+b)/2}^{b} f(x)\, dx.$$

Prove that f is constant on $[0,1]$.

9. Let E be any countable subset of $\mathbb{R}$. Prove that E has measure zero.