Name:________________________________

Math 4633/6633 Section 01 *Practice Final Exam* *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Define a sequence of functions $\{f_n\}$ of real functions by

$$f_n(x) = \frac{\tan^{-1}(nx)}{n} .$$

Prove that each f_n is infinity differentiable and $\{f_n\}$ converges uniformly on $\mathbb{R}$.

2. Consider a sequence of functions $\{f_n\}$, $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{x}{1 + nx^2} .$$

Prove that $f_n \to 0$ uniformly on $\mathbb{R}$ and $\{f_n'\}$ converges pointwise on $\mathbb{R}$ but not uniformly.

3. Let g be a continuous function on $[a, b]$ and $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ such that $\{f_n\}$ converges uniformly to a function f. Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)g(x)\ dx = \int_a^b f(x)g(x)\ dx .$$

4. Let $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ such that

$$f(x) := \sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on $[a, b]$. Prove that

$$\int_a^b f(x)\ dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\ dx .$$

5. Let X be a nonempty set. On $C(X)$ define the following:

$$\|f\|_\infty := \sup_{x\in X} |f(x)| \ .$$

Prove that $(C(X), \|\cdot\|_\infty)$ is a normed liner space.

6. Let X be a nonempty set. On $C^1(X)$ define the following:

$$\|f\|_{\infty,1} = \|f\|_\infty + \|f'\|_\infty := \sup_{x\in X} |f(x)| + \sup_{x\in X} |f'(x)| \ .$$

Prove that $(C^1(X), \|\cdot\|_{\infty,1})$ is a normed liner space.

7. Define $\langle\cdot,\cdot\rangle$ on $C[a,b]$ by

$$\langle f,g\rangle = \int_a^b f(x)g(x)\ dx \ .$$

Prove that $\langle\cdot,\cdot\rangle$ defines an inner product on $C[a,b]$.

8. Let $M_n(\mathbb{R})$ be the set of all $n\times n$ matrices with real entries. Define the map $\langle\cdot,\cdot\rangle : M_n(\mathbb{R})\times M_n(\mathbb{R}) \to \mathbb{R}$ by $\langle A,B\rangle = \mathrm{tr}(B^tA)$. Prove that $\langle\cdot,\cdot\rangle$ defines an inner product on $M_n(\mathbb{R})$.

9. Let $(X,\langle\cdot,\cdot\rangle)$ be an inner product space. Prove that the parallelogram law always holds, that is

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2+\|y\|^2\right) \ .$$