# Math 4683/6683 Practice Exam Solutions

1.) pf: since $-\frac{\pi}{2} \le \tan^{-1}(nx) < \frac{\pi}{2}$ $\forall x \in \mathbb{R}, n \ge 1 \Rightarrow \frac{-\pi}{2n} < \frac{\tan^{-1}(nx)}{n} < \frac{\pi}{2n}$. Then $\frac{\pm\pi}{2n} \to 0$ as $n \to \infty$ independent of $x$. $\therefore$ $\frac{\tan^{-1}(nx)}{n} \to 0$ uniformly on $\mathbb{R}$ by uniform squeeze Th. Moreover by chain rule $f_n'(x) = \frac{1}{1+(nx)^2} \cdot \frac{n}{n} = \frac{1}{1+n^2x^2}$ and induction shows $f_n^{(k)}(x)$ exists. □

2.) pf: Let $t = \sqrt{n}|x|$, then $|h_n(x)| = \frac{1}{\sqrt{n}}\left(\frac{\sqrt{n}|x|}{1+nx^2}\right) = \frac{1}{\sqrt{n}}\left(\frac{t}{1+t^2}\right)$. notice $(1-t)^2 \ge 0$ $\therefore$ $2t \le 1+t^2 \Rightarrow \frac{t}{1+t^2} \le \frac{1}{2}$
$\therefore$ $|h_n(x)| \le \frac{1}{2\sqrt{n}}$ since $0 \le |h_n(x)|$ and $\frac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$ indep. of $x \Rightarrow h_n \to 0$ uniformly on $\mathbb{R}$ by uniform squeeze thm. Next $h_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$ by quotient rule. if $x = 0$ then $h_n'(0) = 1$ and if $x \neq 0$ then $h_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} \le \frac{1}{(1+nx^2)^2} \to 0$ ptwise on $\mathbb{R}$. $\therefore$ if $g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ then $h_n' \to g$ ptwise. but clearly can't go uniformly as $h_n'$ is cont. and $g$ is not. □

3.) pf: Define $h_n(x) = f_n(x)g(x)$. and $h(x) = f(x)g(x)$. Consider $|h_n(x) - h(x)| = |g(x)|\,|f_n(x) - f(x)| \le M|f_n(x) - f(x)|$ since $g$ is cont. on a compact set. But $f_n \to f$ uniformly $\therefore$ $\forall \varepsilon > 0$ pick $\delta > 0$ s.t. $\delta = \frac{\varepsilon}{M}$ s.t. if $n \ge N$ $|f_n(x) - f(x)| < \frac{\varepsilon}{M}$ $\forall x$. $\therefore$ $|h_n(x) - h(x)| \le M\frac{\varepsilon}{M} = \varepsilon$. $\Rightarrow h_n \to h$ uniformly. $\therefore$ by unif. conv. thm. $\lim_{n\to\infty}\int_a^b f_n(x)g(x)\,dx = \lim_{n\to\infty}\int_a^b h_n(x) = \int_a^b \lim_{n\to\infty} h_n(x)\,dx = \int_a^b h(x)\,dx = \int_a^b f(x)g(x)\,dx$ □

4.) Define $g_k(x) = \sum_{n=1}^{k} f_n(x)$. Then $g_k \to f$ uniformly. Thus by uniform conv. Th $\Rightarrow \lim_{k\to\infty}\int_a^b g_k(x)\,dx = \int_a^b f(x)\,dx$
$\Rightarrow \lim_{k\to\infty}\int_a^b \sum_{n=1}^{k} f_n(x)\,dx = \int_a^b f(x)\,dx \Rightarrow \lim_{k\to\infty}\sum_{n=1}^{k}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$
$\Rightarrow \sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. □

5.) pf: since $|f(x)| \ge 0$ $\forall x \Rightarrow \|f\|_\infty \ge 0$. Clearly $\|0\|_\infty = 0$. If $\|f\|_\infty = 0 \Rightarrow |f(x)| = 0$ $\forall x \in X$ $\therefore$ $f \equiv 0$. next $\|\alpha f\|_\infty = \sup_{x\in X}|\alpha f(x)| = |\alpha|\sup_{x\in X}|f(x)| = |\alpha|\,\|f\|_\infty$ $\forall \alpha \in \mathbb{R}$. notice $|f(x)+g(x)| \le |f(x)| + |g(x)|$
$\therefore$ $\|f+g\|_\infty = \sup_{x\in X}|f(x)+g(x)| \le \sup_{x\in X}(|f(x)| + |g(x)|) \le \sup_{x\in X}|f(x)| + \sup_{x\in X}|g(x)| = \|f\|_\infty + \|g\|_\infty$. □

6.) pf: since $|f'(x)|, |f(x)| \ge 0 \Rightarrow \|f\|_{\infty,1} \ge 0$. clearly $\|0\|_{\infty,1} = 0$. If $\|f\|_{\infty,1} = 0 \Rightarrow |f(x)| = 0$ $\forall x \in X$ and $|f'(x)| = 0$ $\forall x \in X$ $\Rightarrow f \equiv 0$. next $\|\alpha f\|_{\infty,1} = \sup_{x\in X}|\alpha f(x)| + \sup_{x\in X}|\alpha f'(x)| = |\alpha|\left(\sup_{x\in X}|f(x)| + \sup_{x\in X}|f'(x)|\right) = |\alpha|\,\|f\|_{\infty,1}$ $\forall \alpha \in \mathbb{R}$.
Notice $|f(x)+g(x)| \le |f(x)+g(x)|$ and $|f'(x)+g'(x)| \le |f'(x)| + |g'(x)|$. $\therefore$ $\|f+g\|_{\infty,1} = \sup_{x\in X}|f(x)+g(x)| + \sup_{x\in X}|f'(x)+g'(x)|$
$\le \sup_{x\in X}(|f(x)| + |g(x)|) + \sup_{x\in X}(|f'(x)| + |g'(x)|) \le (\|f\|_\infty + \|f'\|_\infty) + (\|g\|_\infty + \|g'\|_\infty) = \|f\|_{\infty,1} + \|g\|_{\infty,1}$ □

7.) pf:

$\langle f, f\rangle = \int_a^b f(x)^2 dx \geq 0$ and $\langle 0, 0\rangle = 0$. if $\langle f, f\rangle = 0 \Rightarrow \int_a^b f(x)^2 dx = 0$ since $f(x)^2 \geq 0 \Rightarrow f(x)^2 \equiv 0$ $\forall x \in [a,b]$

$\therefore f \equiv 0$. Next $\langle f, g\rangle = \int_a^b f(x) g(x) dx = \int_a^b g(x) f(x) dx = \langle g, f\rangle$. Finally $\forall \alpha, \beta \in \mathbb{R}$ consider

$\langle \alpha f + \beta g, h\rangle = \int_a^b (\alpha f(x) + \beta g(x)) h(x) dx = \int_a^b \alpha f(x) h(x) dx + \int_a^b \beta g(x) h(x) dx = \alpha \int_a^b f(x) h(x) dx + \beta \int_a^b g(x) h(x) dx =$

$= \alpha \langle f, h\rangle + \beta \langle g, h\rangle$. $\square$

8.) pf:

$\langle A, A\rangle = tr(A^t A) = \sum_{i,j=1}^{n} a_{ij} a_{ij} = \sum_{i,j=1}^{n} a_{ij}^2 \geq 0$. $\langle 0, 0\rangle = tr(0^t 0) = 0$. If $\langle A, A\rangle = 0$

$\Rightarrow \sum_{i,j=0}^{n} a_{ij}^2 = 0 \Rightarrow a_{ij} = 0$ $\forall i, j = 1, \ldots, n$. $\Rightarrow A = 0$. $\langle A, B\rangle = tr(B^t A) = tr((A^t B)^t) = tr(A^t B) = \langle B, A\rangle$.

Finally $\forall \alpha, \beta \in \mathbb{R}$ consider $\langle \alpha A + \beta B, C\rangle = tr((\alpha A + \beta B)^t C) = tr((\alpha A^t + \beta B^t) C) = tr(\alpha A^t C + \beta B^t C) =$

$= \alpha\, tr(A^t C) + \beta\, tr(B^t C) = \alpha \langle A, C\rangle + \beta \langle B, C\rangle$. $\square$

9.) pf:

since $\|x\|^2 = \langle x, x\rangle$ in an inner prod sp. have $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle =$

$= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle + \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle = 2\|x\|^2 + 2\|y\|^2$. $\square$