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MA 4633/6633 Section 01 *Sec. 5.1/5.2 Homework Problems* *August 16, 2018*

1 Section 5.1

1. Show that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$ for all $x \in \mathbb{R}^n$.

2. Show that $\ell^1(\mathbb{N})$ is a proper subset of $\ell^2(\mathbb{N})$ and that if $x \in \ell^1(\mathbb{N})$, then $\|x\|_2 \leq \|x\|_1$. Can you give a more general statement for arbitrary $1 \leq p < \infty$?

3. For $1 \leq p < \infty$, show that $\ell^p(\mathbb{N})$ is an infinite dimensional normed space. HINT: Consider the sequence $\{e_i\}_{i=1}^\infty$ with $e_i = \{\delta_{ij}\}_{j=1}^\infty$ where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

4. Show that $c_{00}(\mathbb{N}) \subset c_0(\mathbb{N}) \subset c(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$.

5. Let $f_k(t) = \dfrac{\sin(kt)}{\sqrt{k}}$ for $0 \leq t \leq 2\pi$. Show that $f_k \to 0$ in $C[0, 2\pi]$ with respect to $\|\cdot\|_\infty$. Does $f_k \to 0$ in $C^1[0, 2\pi]$ with respect to $\|\cdot\|_{\infty,1}$? Why or why not?

6. Let (X, d) be a metric space. If $x_n \to x$ and $y_n \to y$ in X, prove that $d(x_n, y_n) \to d(x, y)$ in $\mathbb{R}$.

2 Section 5.2

7. Let $g \in C[a, b]$ and define

$$E = \{f \in C[a, b] : f(t) < g(t), \text{ for all } t \in [a, b]\} .$$

Prove that E is open in $C[a, b]$ with respect to $\| \cdot \|_\infty$. Is the same statement true if $C[a, b]$ is replaced by $B[a, b]$? Why or why not?

8. Let $g \in C[a, b]$, $r > 0$, and define the following sets:

$$E = \{f \in C[a, b] : f(t) \leq g(t), \text{ for all } t \in [a, b]\}$$
$$F = \{f \in C[a, b] : |f(t)| \leq r, \text{ for all } t \in [a, b]\} .$$

Prove that E and F are closed in $C[a, b]$ with respect to $\| \cdot \|_\infty$.

9. Let (X, d) be a metric space and $E, F \subseteq X$. Prove $\overline{E \cup F} = \overline{E} \cup \overline{F}$. Is it true that $\overline{E \cap F} = \overline{E} \cap \overline{F}$? If so, prove it, otherwise give a counterexample.

10. Let (X, d) be a metric space and $E \subseteq X$. x is said to be a *boundary point* of E if every ε-neighborhood of x contains points of both E and $X \setminus E$. The set of all boundary points of E is denoted by ∂E. Show that ∂E is closed.

11. Let (X, d) be a metric space. Let $F \subseteq X$, prove that F is closed if and only if $F = \overline{F}$.

12. Let $s(\mathbb{N})$ be space of all sequences. Prove that $c_{00}(\mathbb{N})$ is dense in $s(\mathbb{N})$.

13. Let (X, d) be a metric space and $E, F \subseteq X$. Show $E^o \cap F^o = (E \cap F)^o$. Is it true that $E^o \cup F^o = (E \cup F)^o$? If so, prove it, otherwise give a counterexample.

14. Let $(X, \|\cdot\|)$ be a normed space and L a subspace. For $x \in X$, define $\hat{x} = x + L$ be the coset in the quotient space X/L induced by x. Show that $\|\hat{x}\| = \inf\{\|x + l\| : l \in L\}$ defines a seminorm on X/L and that this seminorm is a norm if and only if L is closed.