Name:______________________________

Math 4933/6933 Section 01 *Practice Final Exam* *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Let (S, d) be a metric space. Recall the distance between two subsets $E, F \subseteq S$ is

$$d(E, F) = \inf_{x \in E} d(x, F) \ .$$

Let $F, K \subseteq S$ be closed and compact sets respectively and $F \cap K = \emptyset$. Prove that $d(F, K) > 0$.

2. Let (S_1, d_1) and (S_2, d_2) be two compact metric spaces. Prove that $S_1 \times S_2$ is compact.

3. Let X be a normed vector space and $\mathcal{F} \subseteq C(X)$ an equicontinuous family of functions. Prove that $\overline{\mathcal{F}}$ is equicontinuous.

4. Let X be normed vector space and $\{f_n\} \subseteq C(X)$ is an equicontinuous sequence. Prove that if $f_n \to f$ pointwise on X, then $f_n \to f$ uniformly on X.

5. Let $P[a,b]$ be the space of polynomials on $[a,b]$. Define $P_0[a,b]$ by

$$P_0[a,b] = \{f \in P[a,b] : f(a) = 0\} .$$

Prove that $P_0[a,b]$ is an algebra. Is this set dense in $C[a,b]$? Does $P_0[a,b]$ separate the points of $[a,b]$?

6. Let $P[a,b]$ be the space of polynomials on $[a,b]$. Define $P_1[a,b]$ by

$$P_1[a,b] = \{f \in P[a,b] : f(a) = f(b) = 0\} \ .$$

Prove or disprove that $P_1[a,b]$ separates the points of $[a,b]$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Define $F : C[a,b] \to \mathbb{R}$ by $F(\varphi) = f(\varphi(a))$. Prove that F is differentiable at every $\varphi \in C[a,b]$.

8. Let $P : C^2[0,1] \to C[0,1]$ be given by $P\varphi = \varphi'' - e^{\varphi}$. Compute $dP(\varphi)$.