Name:________________________________

MA 4943/6943 Section 01 *Practice Midterm Exam* *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. Let $\rho : [a, b] \to \mathbb{R}^2$ be a plane curve parametrized in polar coordinates by $\rho = \rho(\theta)$. Prove that the curvature κ of ρ can be expressed as

$$\kappa(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{[(\rho')^2 + \rho^2]^{3/2}}.$$

2. Show that the knowledge of the normal vector $n(s)$ of a curve α, with nonzero torsion everywhere, determines the curvature $\kappa(s)$ and the torsion $\tau(s)$ of α.

3. Assume a regular parametrized curve α has the property that all its tangent lines pass through a fixed point. Prove that the trace (range/image) of α is a (segment of a) straight line. Does the conclusion still hold if α is not regular?

4. Prove that a circle of radius r has constant curvature $\kappa = \dfrac{1}{r}$.

5. Let T^2 be the torus and have the following parametrization:

$$x(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u), \quad 0 < u < 2\pi,\ 0 < v < 2\pi$$

for a fixed a and r. Compute the area of the torus.

6. Suppose S is a surface of revolution parametrized by the following

$$x(u,v) = (u, f(u)\sin v, f(u)\cos v), \quad u \in [a,b],\ v \in [0,2\pi]\ ,$$

where f and is a nonnegative differentiable function. Prove that the area A of S is given by:

$$A = 2\pi \int_a^b f(u)\sqrt{1+(f'(u))^2}\ du\ .$$

7. Suppose S is a surface of revolution parametrized by the following

$$x(u,v) = (f(v)\cos u, f(v)\sin u, g(v)), \quad u \in (0,2\pi),\ v \in [a,b]\ ,$$

where f and g are differentiable functions and $f > 0$. Prove that S is a regular surface and compute its first fundamental form.

8. The sphere, S^2, can be parametrized by the following

$$x(u,v) = (\sin v\cos u, \sin v\sin u, \cos v), \quad u \in (0,2\pi),\ v \in (0,\pi)\ .$$

Prove that S^2 is a regular surface and compute its first fundamental form.

9. Let $C \subset S$be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature κ of C at p satifies

$$|\kappa| \geq \min\{|\kappa_1|, |\kappa_2|\}\ ,$$

where κ_1 and κ_2 are the principal curvatures of S at p.