Name:______________________________

Math 8633 Section 01 | *Practice Final Exam* | *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Does there exist a measurable set E on $[0,1]$ such that for any $0 \leq a < b \leq 1$,

$$m(E \cap [a,b]) = \frac{b-a}{2} \ ?$$

2. Let $\{f_n\}$ be a sequence of nonnegative integrable functions on $\mathbb{R}$ such that

$$\lim_{n\to\infty} f_n(x) = 0$$

almost everywhere. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \min_{1\leq k\leq n} f_k(x) \ dx = 0 \ .$$

3. Let f be a nonnegative integrable function on $[0,1]$ such that

$$\int_0^1 f(x)\ dx = 1\ .$$

Define g by, $g(x) = \sqrt{f(1-x)}$. Prove that g is integrable on $[0,1]$ and

$$\int_0^1 g(x)\ dx \leq 1\ .$$

4. Let f be an integrable function on $[0,1]$ such that for any $0 \leq a < b \leq 1$,

$$\int_a^{(a+b)/2} f(x)\ dx = \int_{(a+b)/2}^b f(x)\ dx\ .$$

Prove that f is constant almost everywhere.

5. Prove that if $f(x)$ is an absolutely continuous on $[0,1]$ and $f'(x) = 1$ almost everywhere, then $f(x) = x + C$ for some constant C.

6. Let f be absolutely continuous on $[a,b]$. Prove that f is Lipschitz on $[a,b]$ if and only if there exists a $c > 0$ such that $|f'| < c$ almost everywhere on $[a,b]$.

7. Let f be a continuous function on $[0,1]$ which is differentiable almost everywhere. Suppose $f' \in L^p[0,1]$ for $p > 1$. Let $\alpha = 1 - \frac{1}{p}$. Prove that there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x-y|^\alpha$$

for all $x, y \in [0,1]$.

8. Let $f \in L^\infty[0,1]$. Prove that for every $p \geq 1$,

$$\exp\left(\int_0^1 f(x)\, dx\right) \leq \|e^f\|_p \,.$$