Name:______________________________

MA 8633 Section 01 | *Practice Exam 1* | *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. Let $A \subseteq [0,1]$ be a measurable set. Define the set

$$B = \cos(A) \equiv \{\cos x : x \in A\} \ .$$

Prove that B is measurable and $m(B) \leq \sin(1) m(A)$.

2. Show there exists measurable sets $X, Y \subseteq \mathbb{R}$ such that $X + Y$ is not measurable. Here

$$X + Y = \{x + y : x \in X, y \in Y\} \ .$$

3. Let $f : [a, b] \to \mathbb{R}$ be a Lipschitz function. Prove that:

- f maps a set of measure zero onto a set of measure zero
- f maps an F_σ set onto an F_σ set
- f maps a measurable set to a measurable set.

4. Let $\{f_n\}$ be a sequence of measurable functions on $(0, 1)$. Prove that

$$E = \{x \in (0, 1) : \{f_n(x)\} \text{ is a convergent sequence}\}$$

is measurable.

5. Let E have measure zero. Prove that if f is a bounded function on E, then f is measurable and

$$\int_E f = 0.$$

6. Prove that if f is bounded and measurable on $[0,1]$, then

$$\lim_{n\to\infty} n \int_0^1 \sin\left(\frac{x}{n}\right) f(x)\, dx = \int_0^1 x f(x)\, dx \ .$$

7. Let f be a bounded measurable function on $[a,b]$. Compute the following limit:

$$\lim_{n\to\infty} \int_a^b \frac{f(x)e^{nx}}{1+e^{nx}}\, dx \ .$$

Be sure to justify your answer.

8. Prove or disprove that the Bounded Convergence Theorem holds for the Riemann integral.