Name:______________________________

Math 8633 Section 01 | *Practice Final Exam* | *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Prove that if $f \in L^1[a,b]$, then

$$\lim_{n\to\infty} n \int_a^b \sin\left(\frac{x}{n}\right) f(x)\, dx = \int_a^b x f(x)\, dx \ .$$

2. Let $E \subseteq \mathbb{R}$ be measurable and $f(x)$ a nonnegative measurable function on E such that the limit

$$\lim_{n\to\infty} \int_E [f(x)]^n\, dx = L \ ,$$

exists and $0 < L < \infty$. Prove that $m\{x \in E : f(x) = 1\} = L$.

3. Let f be an integrable function on $[0,1]$ such that for any $0 \le a < b \le 1$,

$$\int_a^{(a+b)/2} f(x)\, dx = \int_{(a+b)/2}^b f(x)\, dx \ .$$

Prove that f is constant almost everywhere.

4. Does there exist a measurable set E on $[0,1]$ such that for any $0 \le a < b \le 1$,

$$m(E \cap [a,b]) = \frac{b-a}{2} \ ?$$

5. Let $f(x)$ be an integrable function on the interval $[a,b]$. Define recursively the functions:

$$f_0(x) = f(x), \quad f_k(x) = \int_a^x f_{k-1}(t)\, dt, \quad k \ge 1; \quad a \le x \le b \ .$$

Prove that the series

$$F(x) = \sum_{k=1}^{\infty} f_k(x)$$

converges uniformly on $[a,b]$ and that $F(x)$ is an absolutely continuous function on $[a,b]$.

6. Let f be absolutely continuous on $[a, b]$. Prove that f is Lipschitz on $[a, b]$ if and only if there exists a $c > 0$ such that $|f'| < c$ almost everywhere on $[a, b]$.

7. Prove that if $f \in L^{4+\varepsilon}[0, 1]$, where $\varepsilon > 0$, and $g(x) = f(x^2)$, then $g \in L^2[0, 1]$.

8. Let $f \in L^p(E)$ where E is set of finite measure and let $p > r \geq 1$. Prove that $f \in L^r(E)$ and

$$\|f\|_r \leq (mE)^{\frac{1}{r}-\frac{1}{p}} \|f\|_p \ .$$