Name:______________________________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Let f be the cumulative distribution function of a Borel measure ν on the Borel sets of $\mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sqrt{x+1} & \text{if } x \geq 0 \end{cases},$$

that is $\nu(-\infty, x) = f(x)$. If λ is the Lebesgue measure on the Borel sets, find the Lebesgue decomposition of ν with respect to λ. Moreover compute the Radon-Nikodym derivative of the absolutely continuous part of ν with respect to λ.

2. Let $\{\mu_n\}$ be a sequence of measures on a measurable space $(X, \mathcal{M})$ for which there is a constant $c > 0$ such that $\mu_n(X) \leq c$ for all n. Define $\mu : \mathcal{M} \to [0, \infty]$ by

$$\mu = \sum_{n=1}^{\infty} \frac{\mu_n}{2^n} .$$

Show that μ is a measure on $\mathcal{M}$ and that each μ_n is absolutely continuous with respect to μ.

3. Let $\{\mu_n\}$ be a sequence of measures on the Lebesgue measurable space $([a, b], \mathcal{L})$ for which $\{\mu_n([a, b])\}$ is bounded and each μ_n is absolutely continuous with respect to Lebesgue measure m. Show that a subsequence of $\{\mu_n\}$ converges setwise on $\mathcal{M}$ to a measure on $([a, b], \mathcal{L})$ that is absolutely continuous with respect to m.

4. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Prove that $\mathcal{BFA}(X, \mathcal{M}, \mu)$ is a Banach space with respect to $\| \cdot \|_{\text{var}}$.

5. Let h and g be integrable functions on X and Y respectively and define $f(x, y) = h(x)g(y)$. Prove that

$$\int_{X\times Y} f \, d(\mu \times \nu) = \int_X h \, d\mu \int_Y g \, d\nu \ .$$

6. Let $(x, y) \in (-\pi, \pi) \times \mathbb{R}$ and define the following functions:

$$f(x,y) = \begin{cases} \dfrac{\sin x}{|y|} & \text{if } y \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(y) = \int_{-\pi}^{\pi} f(x, y) \, dx \ .$$

Prove that $g(y) \in L^1(\mathbb{R})$. Does it follow that:

$$\int_{\mathbb{R}} \left(\int_{-\pi}^{\pi} f(x, y) \, dx \right) \, dy = \int_{-\pi}^{\pi} \left(\int_{\mathbb{R}} f(x, y) \, dy \right) \, dy \ ?$$

Why or why not?

7. Let X be an uncountable set with the discrete topology. What is $C_c(X)$? What are the Borel subsets of X? Let X^* be the one-point compactification of X. What is $C(X^*)$? What are the Borel subsets of X^*? Prove there is a Borel measure μ on X^* such that $\mu(X^*) = 1$ and

$$\int_X f \, d\mu = 0$$

for each $f \in C_c(X)$.

8. Let X be a compact Hausdorff space and μ a Borel measure on $\mathcal{B}(X)$. Show that there is a constant $c > 0$ such that

$$\left| \int_X f \, d\mu \right| \leq c\|f\|_\infty$$

for all $f \in C(X)$.