Name:______________________________

MA 8643 Section 01 | *Practice Midterm Exam* | *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. Show that it is not possible to express a compact interval of real numbers as the pairwise disjoint union of a countable collection (having more than one member) of compact intervals.

2. Show that the arbitrary collection of Tychonoff spaces, with the product topology, is also Tychonoff.

3. Let X be a topological space. Prove that X is countable compact if and only if whenever $\{F_n\}$ is a decreasing sequence of nonempty closed subsets of X, the intersection

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset \ .$$

4. Let $(X, \mathcal{A})$ be a measurable space and let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Define

$$\nu = \mu_1 - \mu_2 \ .$$

If one of μ_i, $i = 1, 2$, is finite, prove that ν is a signed measure on $(X, \mathcal{A})$.

5. Let ν be a signed measure on some measurable space. Prove that if E is any measurable set, then

$$-\nu^{-}(E) \leq \nu(E) \leq \nu^{+}(E) \text{ and } |\nu(E)| \leq |\nu|(E) \ .$$

6. Let η be the counting measure on $\mathbb{Z}$. Characterize the nonnegative real-valued functions that are integrable over $\mathbb{Z}$ with respect to η and the value

$$\int_{\mathbb{Z}} f \ d\eta \ .$$

7. Suppose f and g are nonnegative measurable functions on X for which f^2 and g^2 are integrable over X with respect to μ. Show that fg is integrable over X with respect to μ.

8. Let $\nu : \mathcal{M} \to [0, \infty)$ be a finite additive set function. Show that if f is a bounded measurable function on X, then the integral of f over X with respect to ν, $\int_X f d\nu$, can be defined so that

$$\int_X \chi_E \ d\nu = \nu(E) \ ,$$

if E is measurable and integration is linear, monotone, and additive over domains for bounded measurable functions.

9. Let $\mathcal{S}$ be an algebra of subsets of a set X. We say that a function $\varphi : X \to \mathbb{R}$ is $\mathcal{S}$-simple provided

$$\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$$

where each $A_k \in \mathcal{S}$. Let μ be a premeasure on $\mathcal{S}$ and $\overline{\mu}$ its Carathéodory extension. Given $\varepsilon > 0$ and a function f that integrable over X with respect to $\overline{\mu}$, show there is an $\mathcal{S}$-simple function φ such that

$$\int_X |f - \varphi| \ d\overline{\mu} < \varepsilon \ .$$