Name:______________________________



Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

**1**. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\nu : \mathcal{M} \to [0, \infty)$ a finitely additive set function with the property that for each $\varepsilon > 0$, there is a $\delta > 0$ such that for a measurable set $E$, if $\mu(E) < \delta$, then $\nu(E) < \varepsilon$. Show that $\nu$ is a measure on $\mathcal{M}$.

**2**. Let $(X, \mathcal{M}$ be a measurable space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$ that converges setwise on $\mathcal{M}$ to $\nu$. Assume $\nu(X) < \infty$. Let $\{E_k\}$ be a descending sequence of measurable sets with empty intersection. Show that for each $\varepsilon > 0$, there is a natural number $k$ for which $\nu_n(E_k) < \varepsilon$ for all $n$.

**3**. Let $\{\mu_n\}$ be a sequence of measures on the Lebesgue measurable space $([a, b], \mathcal{L})$ for which $\{\mu_n([a, b])\}$ is bounded and each $\mu_n$ is absolutely continuous with respect to Lebesgue measure $m$. Show that a subsequence of $\{\mu_n\}$ converges setwise on $\mathcal{M}$ to a measure on $([a, b], \mathcal{L})$ that is absolutely continuous with respect to $m$.

**4**. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Prove that $\mathcal{BFA}(X, \mathcal{M}, \mu)$ is a Banach space with respect to $\| \cdot \|_{\text{var}}$.

**5**. Let $h$ and $g$ be integrable functions on $X$ and $Y$ respectively and define $f(x, y) = h(x)g(y)$. Prove that

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X h \, d\mu \int_Y g \, d\nu \, .$$

**6**. Let $(x, y) \in (-\pi, \pi) \times \mathbb{R}$ and define the following functions:

$$f(x,y) = \begin{cases} \dfrac{\sin x}{|y|} & \text{if } y \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(y) = \int_{-\pi}^{\pi} f(x,y)\, dx\,.$$

Prove that $g(y) \in L^1(\mathbb{R})$. Does it follow that:

$$\int_{\mathbb{R}} \left( \int_{-\pi}^{\pi} f(x,y)\, dx \right) dy = \int_{-\pi}^{\pi} \left( \int_{\mathbb{R}} f(x,y)\, dy \right) dy\,?$$

Why or why not?

**7**. Let $X$ be an uncountable set with the discrete topology. What is $C_c(X)$? What are the Borel subsets of $X$? Let $X^*$ be the one-point compactification of $X$. What is $C(X^*)$? What are the Borel subsets of $X^*$? Prove there is a Borel measure $\mu$ on $X^*$ such that $\mu(X^*) = 1$ and

$$\int_X f \ d\mu = 0$$

for each $f \in C_c(X)$.

**8**. Let $k(x, y)$ be a bounded Borel measurable function on $X \times Y$, and let $\mu$ and $\nu$ be Radon measures on $X$ and $Y$ respectively. Prove that

$$\begin{aligned} \int_{X\times Y} k(x,y)\varphi(x)\psi(y)\, d(\mu \times \nu) &= \int_Y \left( \int_X k(x,y)\varphi(x)\, d\mu \right) \psi(y)\, d\nu \\ &= \int_X \left( \int_Y k(x,y)\psi(y)\, d\nu \right) \varphi(x)\, d\mu \end{aligned}$$

for all $\varphi \in C_c(X)$ and $\psi \in C_c(Y)$. Moreover show that if the integral in the above equation is zero for all $\varphi \in C_c(X)$ and all $\psi \in C_c(Y)$, then $k = 0$ a.e. $\mu \times \nu$.