Name:______________________________

Math 8663 Section 01 *Practice Final Exam* *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Let $\{x_n\}$ br a sequence in $\ell^1(\mathbb{N})$. Prove that

$$\sum_{k=1}^{\infty} x_{n,k} y_k \to 0 \text{ as } n \to \infty$$

for every $y \in c_0$ if and only if $\sup \{\|x_n\|_1 : n \geq 1\} < \infty$ and $x_{n,k} \to 0$ as $n \to \infty$ for every $k \geq 1$.

2. Let (X, Ω, μ) be a σ-finite measure space and $\{f_n\} \in L^1(\Omega)$. Show that

$$\int_{\Omega} f_n g \, d\mu \to 0 \text{ as } n \to \infty$$

for every $g \in L^{\infty}(\Omega)$ if and only if $\sup \{\|f_n\|_1 : n \geq 1\} < \infty$ and

$$\int_E f_n \, d\mu \to 0 \text{ as } n \to \infty$$

for every $E \subset \Omega$.

3. Let X be a Banach space and $A \in L(X)$. Suppose A has the property that $A^n = 0$ for some n. Prove that $\sigma(A) = \{0\}$.

4. Let X be a compact space and $f \in C(X)$. Prove that $\sigma(f) = f(X)$.

5. Define the following Hilbert spaces:

$$\mathcal{H}_1 = \left\{ x \in \ell^2(\mathbb{N}) : \sum_{k=1}^{\infty} \frac{1}{k^2} |x_k|^2 < \infty \right\} \text{ and } \mathcal{H}_2 = \left\{ x \in \ell^2(\mathbb{N}) : \sum_{k=1}^{\infty} \frac{1}{k^3} |x_k|^2 < \infty \right\}$$

These are so-called weighted ℓ^2-spaces. Let c be a real number such that $0 < c \leq 1$ and define the following bounded operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ by:

$$(Tx)_k = \sum_{n=1}^{k} \frac{c^{k-1-n}}{n^2} x_n$$

Prove that T is a compact operator.

6. Define the following operator on $L^2[-\pi, \pi]$ by:

$$A = -\frac{d^2}{dx^2}$$

whose domain is

$$\begin{aligned} \mathcal{D} &= \mathrm{dom}(A) \\ &= \left\{ f \in C^2(-\pi, \pi) \cap C[-\pi, \pi] : f', f'' \in L^2[-\pi, \pi] \ , \ f(-\pi) = f(\pi), \ f'(-\pi) = f'(\pi) \right\} \end{aligned}$$

Prove that A is an unbounded self-adjoint operator. Moreover compute $\sigma(A)$ and classify the point, continuous and residual spectra.

7. Let $\mathcal{H}$ be a Hilbert space and let A be a self-adjoint(not necessarily bounded) operator defined on $\mathcal{H}$ with densely defined domain in $\mathcal{H}$. Prove that

$$\lim_{t \to \infty} e^{-tA^2} = 0$$