Name:________________________________

Math 8673 Section 01 | *Practice Final Exam* | *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

**1**. Using the Fourier transform, solve the following initial value problem:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

**2**. Using the Fourier transform, for a fixed $\lambda$ and $f \in L^2(\mathbb{R}^n)$ solve the problem:

$$\Delta u - \lambda u = f \quad \text{in } \mathbb{R}^n$$

**3**. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the bounded operators of $\mathcal{H}$. Let $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ be the compact operators. Prove that $\mathcal{B}(\mathcal{H})$ is a $C^*-$algebra. Moreover, prove that $\mathcal{K}$ is an ideal and a hereditary subalgebra of $\mathcal{B}(\mathcal{H})$.

**4**. Find all the left ideals and hereditary subalgebras of $M_{nn}(\mathbb{C})$.

**5**. Let $\mathcal{A}$ be a seperable $C^*-$algebra and $\{\varphi_n\}$ a weak$^*$ dense sequence in the state space. Define

$$\varphi = \sum_n \frac{1}{2^n}\varphi_n$$

Prove that $\varphi$ is a state and the representation $\pi_\varphi$ is faithful.

**6**. A **Banach limit**, denoted LIM, is a continuous linear functional $\psi : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ such that for all $x = \{x_n\}$ and $y = \{y_n\}$ in $\ell^\infty(\mathbb{N})$ and complex number $\alpha$ we have:

(a) If $x_n \geq 0$ for all $n$, then $\psi(x) \geq 0$

(b) $\psi(x) = \psi(Sx)$ where $(Sx)_n = x_{n+1}$, i.e. $S$ is the shift operator

(c) If $x$ is a convergent sequence, then $\psi(x) = \lim x$

Fix a Banach limit LIM on $\ell^\infty(\mathbb{N})$ and let $\{e_n\}$ be an orthonormal basis for some seperable Hilbert space $\mathcal{H}$. Define $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by $\varphi(A) = \mathrm{LIM}\langle Ae_n, e_n\rangle$. Prove that $\varphi$ is a state on $\mathcal{B}(\mathcal{H})$. If $(\pi_\varphi, \mathcal{H}_\varphi)$ is the representation defined via the GNS construction, show that $\mathrm{Ker}(\pi_\varphi) = \mathcal{K}$ where $\mathcal{K}$ is the ideal of compact operators in $\mathcal{B}(\mathcal{H})$.

**7**. Let $T$ be a closed, not necessarily bounded operator. Suppose $\rho(T) \neq \emptyset$ and $R_T(\lambda)$ is a compact operator for some $\lambda \in \rho(T)$. Prove that $T$ is a Fredholm operator with compact parametrix.

**8**. Let $D$ be a bounded region in $\mathbb{C}$ such that $\partial D = \partial \overline{D}$, i.e. $D$ has closed boundary. Let $M_z$ be the multiplication by $z$ operator defined on $L^2(D)$. Prove that $\sigma_{le}(M_z) \cap \sigma_{re}(M_z) = \partial D$ and $\mathrm{ind}(M_z - \lambda) = -1$ for $\lambda \in D$. Moreover show that if $\lambda \notin \overline{D}$, then $M_z - \lambda$ is invertible.