Analysis of *p*-Adic Numbers

Matt McBride

University of Oklahoma

March 8, 2013

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• Let $x \in \mathbb{Q}$, $x \neq 0$, then

$$x = \pm p^{\gamma} \frac{a}{b}$$

for $\gamma \in \mathbb{Z}$, $a, b \in \mathbb{Z}_+$, with a, b not divisble by p and $\gcd(a, b) = 1$

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- ▶ *p*-adic norm: $|x|_p$ of $x \in \mathbb{Q}$ is $|x|_p = p^{-\gamma}$ for $x \neq 0$, $|0|_p := 0$
- ▶ Q_p: field of p-adic numbers, is the completion of the field Q w.r.t. the norm | · |_p

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Facts and Examples

Every nonzero p-adic number, can be written as

$$x=\sum_{i=0}^{\infty}x_ip^i,$$

where $x_i = 0, 1, 2, \dots, p - 1$, $x_0 \neq 0$

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 −γ is called the order of x and denoted ord x = −γ and ord 0 := −∞

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If x a nonzero integer, then

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$$x = \sum_{i=0}^{n} x_i p^i$$

Norm Properties

▶ 1.) $|x|_{p} \ge 0$, 2.) $|xy|_{p} \le |x|_{p}|y|_{p}$

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$$|x + y|_p \le \max(|x|_p, |y|_p)$$
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$$\blacktriangleright |x+y|_p = \max(|x|_p, |y|_p) \text{ iff } |x|_p \neq |y|_p$$

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• Means the norm in \mathbb{Q}_p is an ultrametric

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► $B_{\gamma}(a) = \{x \in \mathbb{Q}_p : |x - a|_p \le p^{\gamma}\}$ -disk with center a and radius p^{γ}

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Topology of \mathbb{Q}_p

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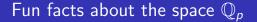
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- Have the following relations:

$$egin{aligned} B_\gamma(a) &= igcup_{\gamma' \leq \gamma} S_{\gamma'}(a), \quad S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a) \ \mathbb{Q}_p &= igcup_{\gamma \in \mathbb{Z}} B_\gamma(a), \quad \mathbb{Q}_p^ imes = igcup_{\gamma \in \mathbb{Z}} S_\gamma(a) \end{aligned}$$

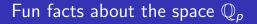
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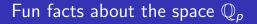
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- A disk is open and compact

Some Calculus of \mathbb{Q}_p

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- We have the following integrals:

$$\int_{\mathbb{Z}_p} d_p x = 1, \quad \int_{B_{\gamma}} d_p x = p^{\gamma}, \quad \int_{S_{\gamma}} d_p x = \left(1 - \frac{1}{p}\right) p^{\gamma}$$
$$\int_{\mathbb{Q}_p} f(x) d_p x = \sum_{\gamma = -\infty}^{\infty} \int_{S_{\gamma}} f(x) d_p x, \quad \int_{B_{\gamma}} |x|_p^{\alpha - 1} = \frac{1 - p^{-1}}{1 - p^{-\alpha}} p^{\alpha \gamma}, \, \alpha > 0$$

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The goal is to find functions u that satisfy this type of boundary value problem.

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- ► This has been studied very extensively classically, meaning when $L = \frac{d}{dx}$ or

$$L = a_n(x)\frac{d^n}{dx^n} + \cdots + a_1(x)\frac{d}{dx} + a_0(x),$$

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These types of questions will lead to problems in ODEs or PDEs. Current research still goes on in this type of question.

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- The boundary of Ω , $\partial \Omega$ becomes Cantor Sets!
- Now L becomes some kind of difference of the vertices and edges making it seem like the problem becomes a discrete problem.

Image: A matrix

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- Now L becomes some kind of difference of the vertices and edges making it seem like the problem becomes a discrete problem.
- ► Now one has to study function spaces of sequences and series that relate to Q_p



Vladimirov, V.S., Table of Integrals of Complex-valued Functions of *p*-Adic Arguments, arXiv:math-ph/9911027v1 22 Nov 1999.

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