Dirac operators on the quantum punctured disk

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Parametrices to the Dirac operators

Method of proofs(Commutative and Quantum)

Bibliography

Operator and Hilbert space

- Let the punctured disk be:

$$\mathbb{D}^* = \{z \in \mathbb{C} \ : \ 0 < |z| \leq 1\} \simeq \mathbb{R}_+ \times S^1$$

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- and Hilbert space $L^2(\mathbb{D}^*, d\mu)$ where

$$d\mu(z) = \frac{1}{2i|z|^2} dz \wedge d\overline{z}$$

APS conditions

- Let $P_{\geq 0}$ be the othogonal projection onto $\mathrm{span}\,\{e^{in\varphi}\}_{n\geq 0}$. Let $z = re^{i\varphi}$ and write $f(z) = f(r, \varphi)$ on $\mathbb{D}^*$.

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- APS conditions:

$$\mathrm{dom}(D) = \left\{f \in L^2(\mathbb{D}^*, d\mu) \ : \ Df \in L^2(\mathbb{D}^*, d\mu), \ \ P_{\geq 0}f(1,\cdot) = 0\right\}$$

Dirac operator decomposition

- Using the polar coordinates

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- Along with the projection, $f(z)$ has a Fourier decomposition and

$$f(z) = \sum_{n\in\mathbb{Z}} f_n(r)e^{-in\varphi}$$

$$Df(z) = \sum_{n\in\mathbb{Z}} \left(-rf_n'(r) - nf_n(r)\right) e^{-in\varphi}$$

Operator and Hilbert space I

- Let $\{e_k\}$ be the canonical basis for $\ell^2(\mathbb{Z})$. $Ue_k = e_{k+1}$, the bilateral shift, $Ke_k = ke_k$, the label operator then by the functional calculus if $f : \mathbb{Z} \to \mathbb{C}$, then $f(K)$ is diagonal and $f(K)e_k = f(k)e_k$.

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- Let $\{w(k)\}$ be a sequence of real numbers:
-
  1. $w(k) < w(k+1)$
  2. $\lim\limits_{k\to\infty} w(k) =: w_+ \text{ exists}$
  3. $\lim\limits_{k\to-\infty} w(k) = 0$
  4. $\sup\limits_{k} \dfrac{w(k)}{w(k-1)} < \infty$

Operator and Hilbert space II

- $w : \mathbb{Z} \to \mathbb{C}$ gives the diagonal operator $w(K)$ and $U_w e_k := Uw(K)e_k = w(k)e_{k+1}$. Let $S := [U_w^*, U_w]$ and $\mathrm{tr}(S) = w_+^2$

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- Quantum punctured disk is $C^*(U_w)$ and

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(U_w) \xrightarrow{\sigma} C(S^1) \longrightarrow 0$$

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- $\mathcal{K}$ is ideal of compact operators, σ is the noncommutative "restriction to the boundary" map.

Operator and Hilbert space III

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$$\mathcal{H} = \overline{(C^*(U_w), \langle \cdot, \cdot \rangle_\tau = \| \cdot \|_w^2)}$$

and $\|b\|_w^2 = \tau(bb^*)$

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  and $\|b\|_w^2 = \tau(bb^*)$
- The quantum Dirac operator acting in $\mathcal{H}$

$$Db = -S^{-1} U_w^* [b, U_w]$$

APS

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- APS conditions:

$$\mathrm{dom}(D) = \left\{ b \in \mathcal{H} \ : \ \|Db\|_w^2 < \infty, \ \ P_{\geq 0}\sigma(b) = 0 \right\}$$

Dirac operator decomposition I

- Partial Fourier decomposition $\mathcal{H} \simeq \bigoplus_{n\in\mathbb{Z}} \ell^2_a(\mathbb{Z})$ and $a(k) := w(k)^2/S(k)$.

Dirac operator decomposition I

- Partial Fourier decomposition $\mathcal{H} \simeq \bigoplus_{n\in\mathbb{Z}} \ell_a^2(\mathbb{Z})$ and $a(k) := w(k)^2/S(k)$.
- $$b = \sum_{n\in\mathbb{Z}} g_n(K)\,(U^*)^n$$

Dirac operator decomposition II

$$Db = \sum_{n>0} \overline{A}^{(n)} g_n(K)(U^*)^n + \sum_{n\leq 0} \overline{A_0}^{(n)} g_n(K)(U^*)^n$$

where

$$\overline{A}^{(n)} g(k) = a(k)(g(k) - c^{(n)}(k)g(k+1))$$

and

$$\mathrm{dom}(\overline{A}) = \left\{ g \in \ell^2_a(\mathbb{Z}) \ : \ \|\overline{A}g\|_a < \infty \right\}.$$

Additionally consider the operator $\overline{A_0}^{(n)}$ which is the operator $\overline{A}^{(n)}$ but with domain

$$\mathrm{dom}(\overline{A_0}^{(n)}) = \{ g \in \mathrm{dom}(\overline{A}^{(n)}) : g_\infty := \lim_{k\to\infty} g(k) = 0 \}.$$

Classical Case: Parametrix to the Dirac operator

- The parametrix to the classical Dirac operator: $Q = \oplus_{n\in\mathbb{Z}} Q^{(n)}$

Classical Case: Parametrix to the Dirac operator

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- $$Q^{(n)}g_n(r) = \begin{cases} -\int_0^r \left(\frac{\rho}{r}\right)^n g_n(\rho)\frac{d\rho}{\rho} & n > 0 \\ \int_r^1 \left(\frac{\rho}{r}\right)^n g_n(\rho)\frac{d\rho}{\rho} & n \leq 0 \end{cases}$$

Non-Commutative Case: Parametrix to the quantum Dirac operator

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Q^{(n)}g(k) = -\sum_{l<k} \frac{S(l)}{w(l)^2} g(l) \qquad \text{for } n = 0
$$

$$
Q^{(n)}g(k) = -\sum_{l<k} \frac{w(l)\cdots w(l+n-1)}{w(k)\cdots w(k+n-1)} \cdot \frac{S(l)}{w(l)^2} g(l) \qquad \text{for } n > 0
$$

$$
Q^{(n)}g(k) = \sum_{k\leq l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)} \cdot \frac{S(l)}{w(l)^2} g(l) \qquad \text{for } n < 0
$$

Schur-Young Inequality

Lemma

(Schur-Young Inequality) Let $T : L^2(Y) \longrightarrow L^2(X)$ *be an integral operator:*

$$Tf(x) = \int K(x,y)f(y)dy$$

Then one has

$$\|T\|^2 \leq \left(\sup_{x \in X} \int_Y |K(x,y)|dy \right) \left(\sup_{y \in Y} \int_X |K(x,y)|dx \right).$$

Method of proof: commutative case I

- For $n < 0$

$$Q^{(n)} g_n(r) = \int_0^1 K(r,\rho) g_n(\rho) \frac{d\rho}{\rho}$$

Method of proof: commutative case I

- For $n < 0$
$$Q^{(n)}g_n(r) = \int_0^1 K(r,\rho)g_n(\rho)\frac{d\rho}{\rho}$$
-
$$K(r,\rho) = \chi\left(\frac{r}{\rho}\right)\left(\frac{r}{\rho}\right)^{|n|}$$

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- For $n < 0$
$$Q^{(n)}g_n(r) = \int_0^1 K(r,\rho)g_n(\rho)\frac{d\rho}{\rho}$$
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$$K(r,\rho) = \chi\left(\frac{r}{\rho}\right)\left(\frac{r}{\rho}\right)^{|n|}$$
-
$$\chi(t) = \begin{cases} 1 & \text{for } t \leq 1 \\ 0 & \text{else} \end{cases}$$

Method of proof: commutative case II

- Using the Schur-Young Inequality

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$$\|Q^{(n)}\| \leq \frac{1}{|n|}$$

Method of proof: commutative case II

- Using the Schur-Young Inequality
- $$\|Q^{(n)}\| \leq \frac{1}{|n|}$$
- Similarly for $n > 0$ the above holds. The $n = 0$ term is ignored.

Method of proof: quantum case I

- For $n < 0$

$$Q^{(n)}g(k) = \sum_{l \in \mathbb{Z}} K(l, k) \frac{S(l)}{w(l)^2} g(l)$$

Method of proof: quantum case I

- For $n < 0$

$$Q^{(n)}g(k) = \sum_{l\in\mathbb{Z}} K(l,k)\frac{S(l)}{w(l)^2}g(l)$$

-

$$K(l,k) = \chi\left(\frac{k}{l}\right)\frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)}$$

Method of proof: noncommutative case II

- Using the Schur-Young Inequality and Riemann sum estimates

Method of proof: noncommutative case II

- Using the Schur-Young Inequality and Riemann sum estimates
- $$\|Q^{(n)}\|_a^2 \leq 2\left(\sup_l \frac{w(l)}{w(l-1)}\right) < \infty$$

Method of proof: noncommutative case II

- Using the Schur-Young Inequality and Riemann sum estimates
- $$\|Q^{(n)}\|_a^2 \le 2\left(\sup_l \frac{w(l)}{w(l-1)}\right) < \infty$$
- Similarly for $n > 0$ the above holds. The $n = 0$ term is ignored.

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Thank You